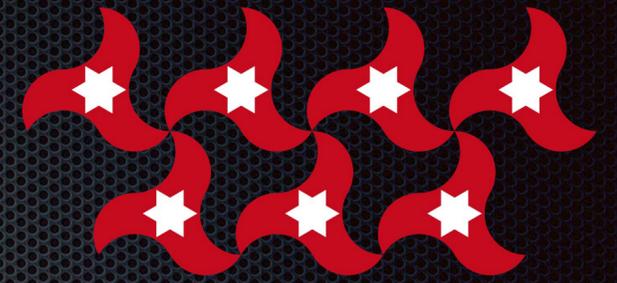




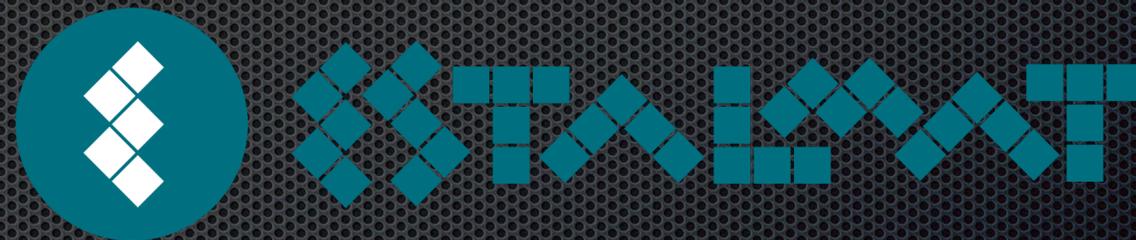
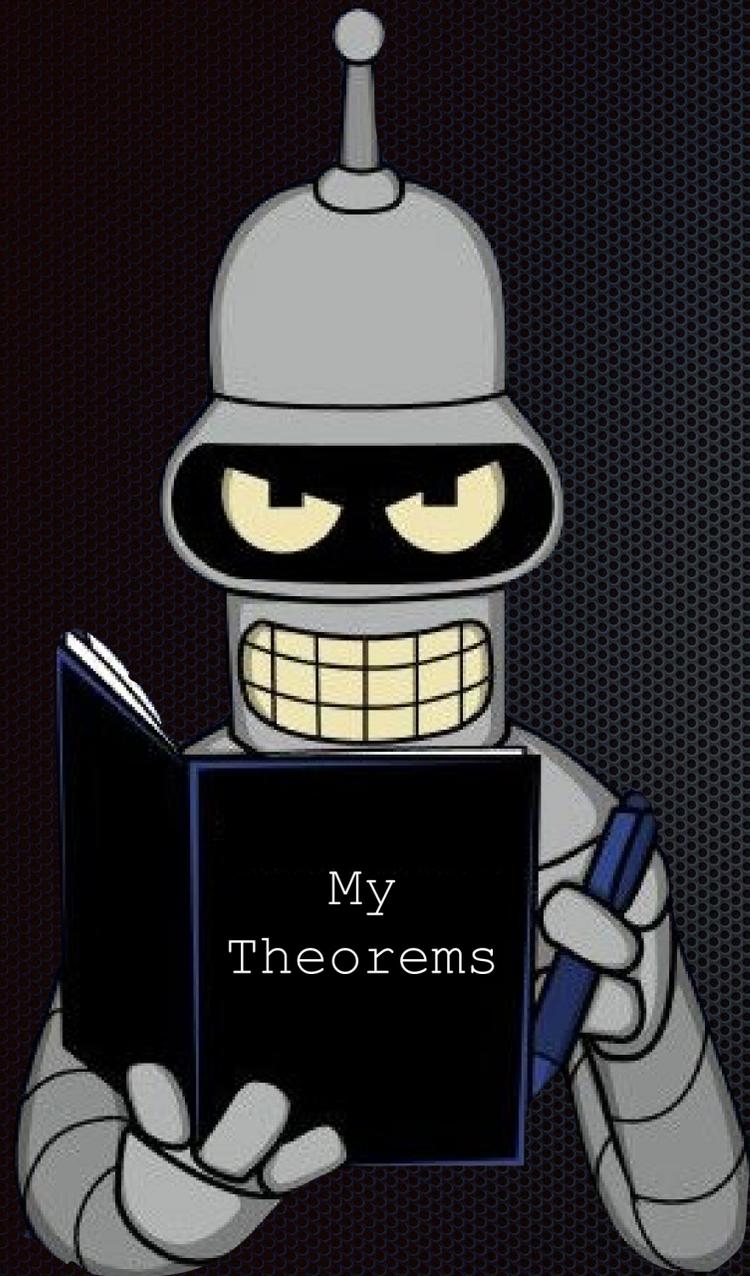
ESTALMAT

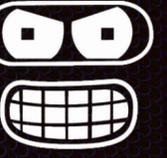


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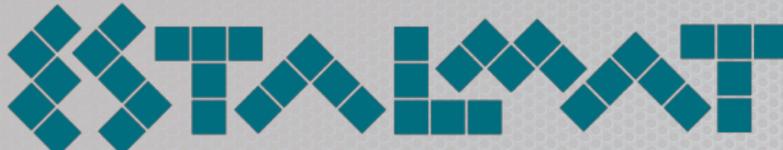
El Teorema de Futurama

Angélica Benito y Ana Granados





De bien nacidos es ser agradecidos...



ESTÍMULO DEL TALENTO MATEMÁTICO



REAL ACADEMIA DE CIENCIAS EXACTAS, FÍSICAS Y NATURALES DE ESPAÑA



GOBIERNO DE ESPAÑA



MINISTERIO DE CIENCIA E INNOVACIÓN



FECYT INNOVACIÓN



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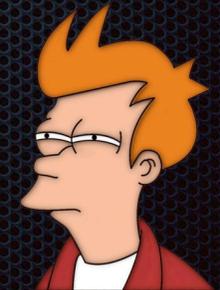


Primera parte: ¡permutemos!

- ¿Qué es una permutación?



- ¿Conoces algún ejemplo de permutación que no sea de números?



- Dados 4 objetos distintos, ¿cuántas posibles permutaciones hay?
¿Y si tenemos 7? ¿Y para n objetos distintos?



Primera parte: ¡denotemos!

Empecemos con un ejemplo:

1 → 4

3 → 1

4 → 3

Lo escribimos como una **matriz** de dos filas:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$



Notación de **ciclos**: Escribimos (1 4 3). ¿Qué quiere decir?

El 1 va al 4 que va al 3 que de vuelta va al 1, formando un ciclo.



Primera parte: ¡preguntemos!

1. ¿Cualquier permutación de cuatro elementos se puede escribir como un ciclo?



2. ¿Cómo podemos resolver este problema? Fíjate en el ejemplo que has pensado, ¿qué harías?



3. **Fijamos la notación de ciclos:** Empezamos por el elemento más pequeño que no se queda fijo hasta completar con cada ciclo.



Primera parte: ¡combinemos!

¿Qué pasa si queremos combinar varias permutaciones? ¿En qué situaciones de la vida real hacemos esto?

Un **ejemplo**: primero permutamos $(1\ 2)$ y a continuación $(1\ 3)$

$$(1\ 2)(1\ 3) = ?$$

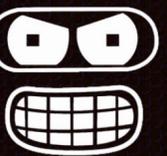
Propiedades de las permutaciones:

- ¿Conmutativo? ¿ $(1\ 2)(1\ 3) = (1\ 3)(1\ 2)$?
- ¿Existe identidad?
- ¿Existe inversa? ¿cómo se construye?
- ¿Asociativo?

¡Estructura de grupo!



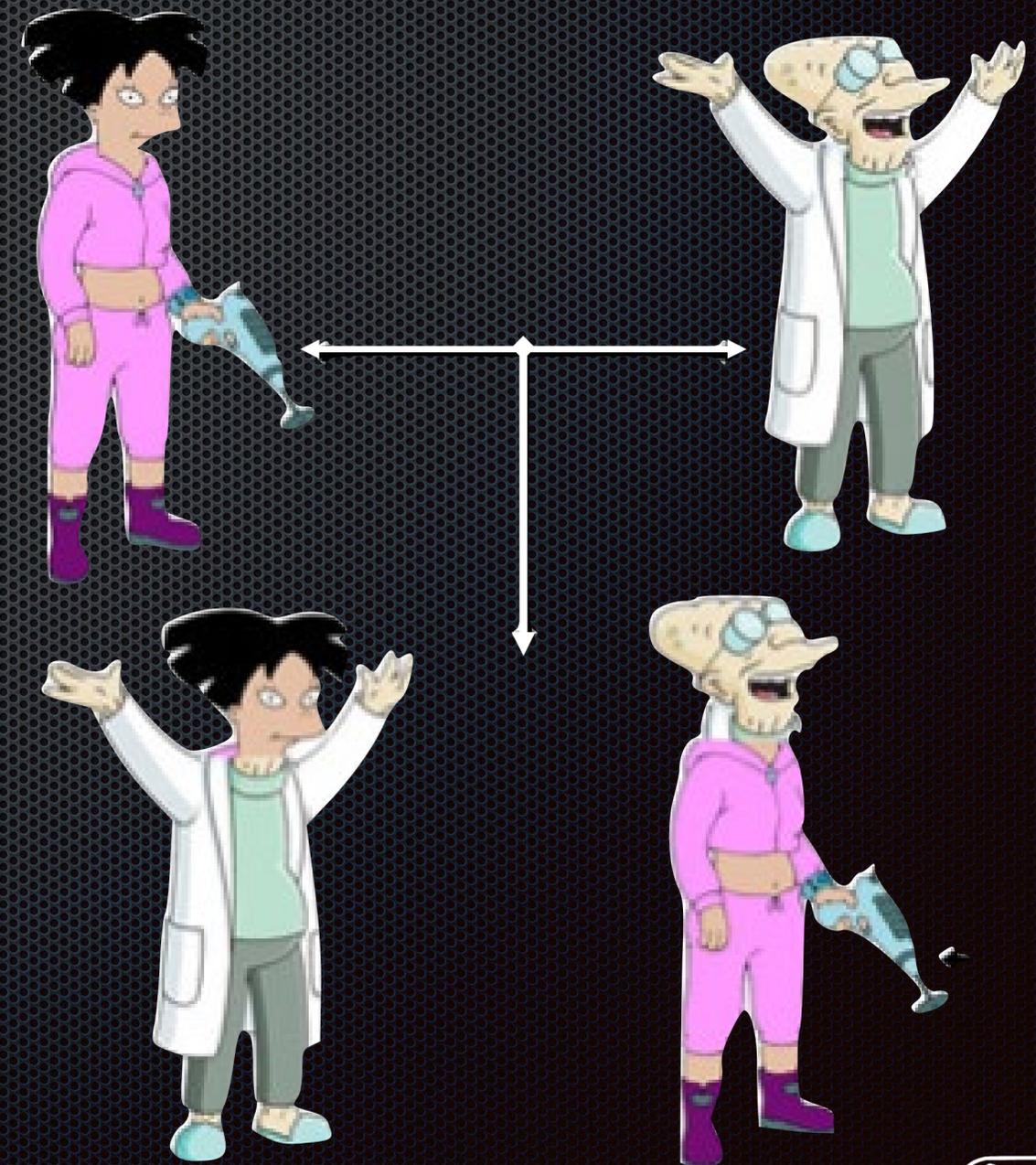
Segunda parte: ¡El Teorema de Futurama!



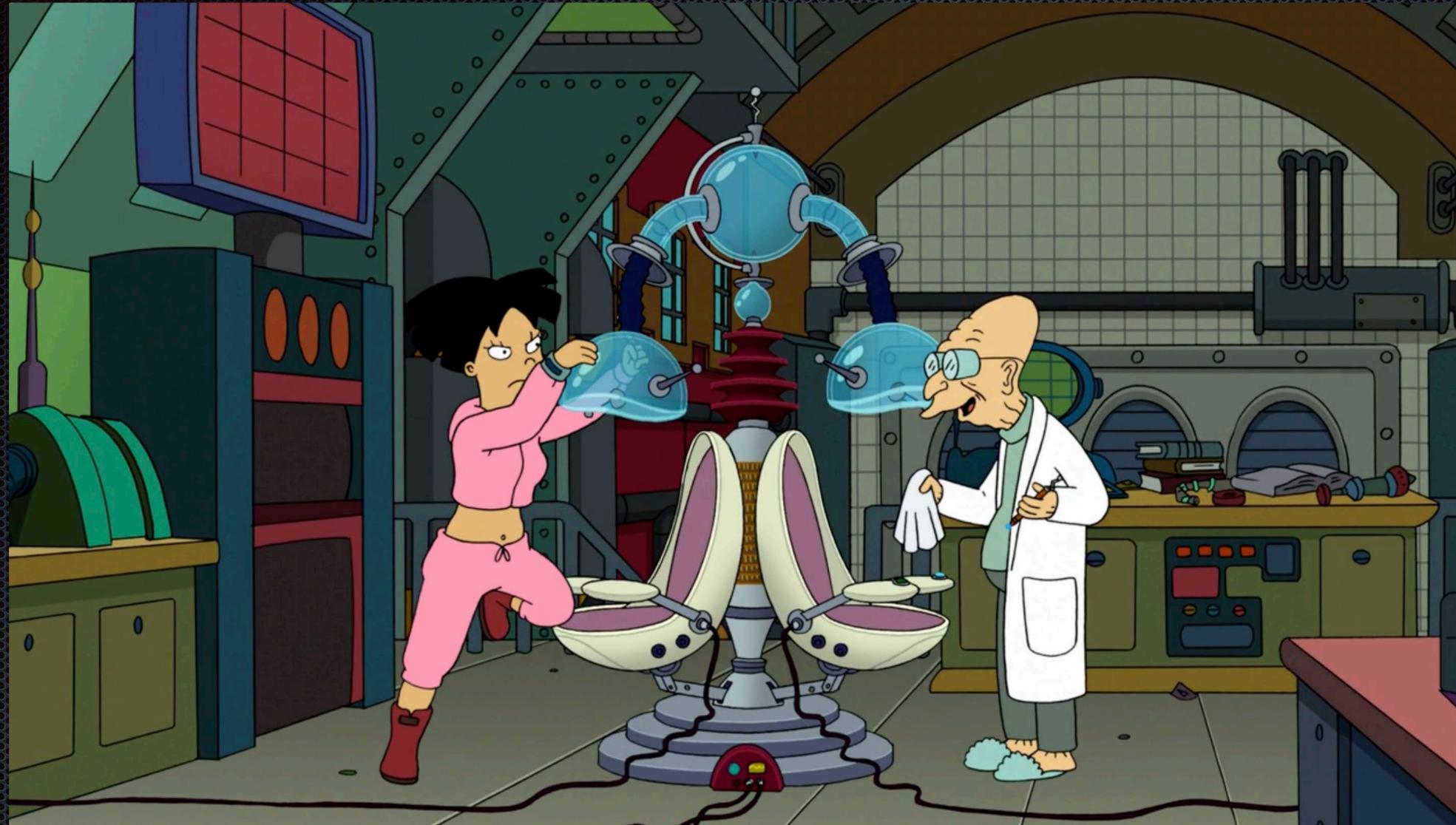
Segunda parte: El problema

El Profesor Farnsworth y Amy deciden probar su flamante “**máquina intercambiadora de mentes**” que acaban de inventar.

Cuando intentan volver a su cuerpo original, descubren un **error** en el diseño de la máquina: un mismo par de cuerpos **no** puede entrar en la máquina más de una vez.

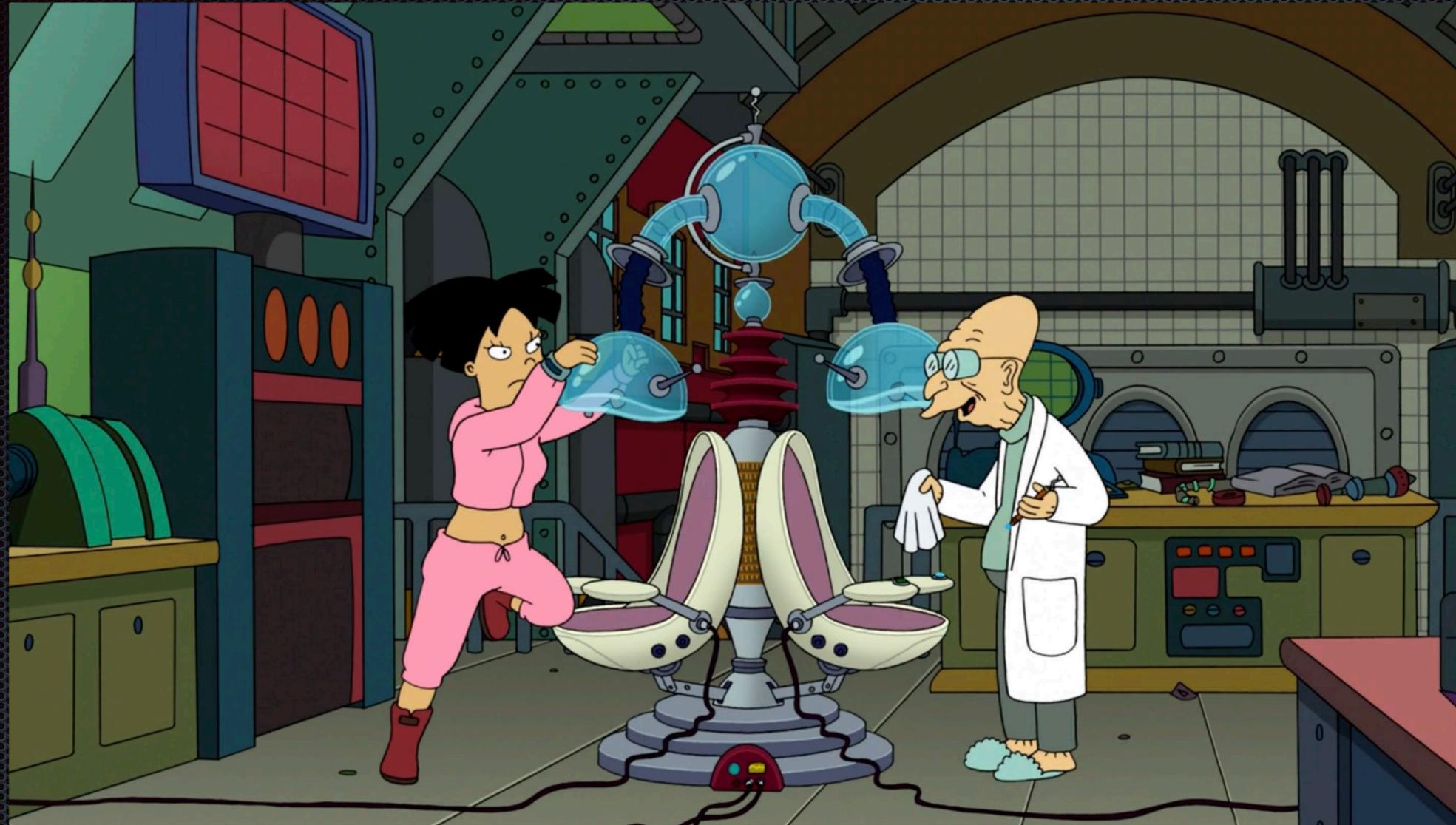


Segunda parte: El vídeo





Segunda parte: El vídeo

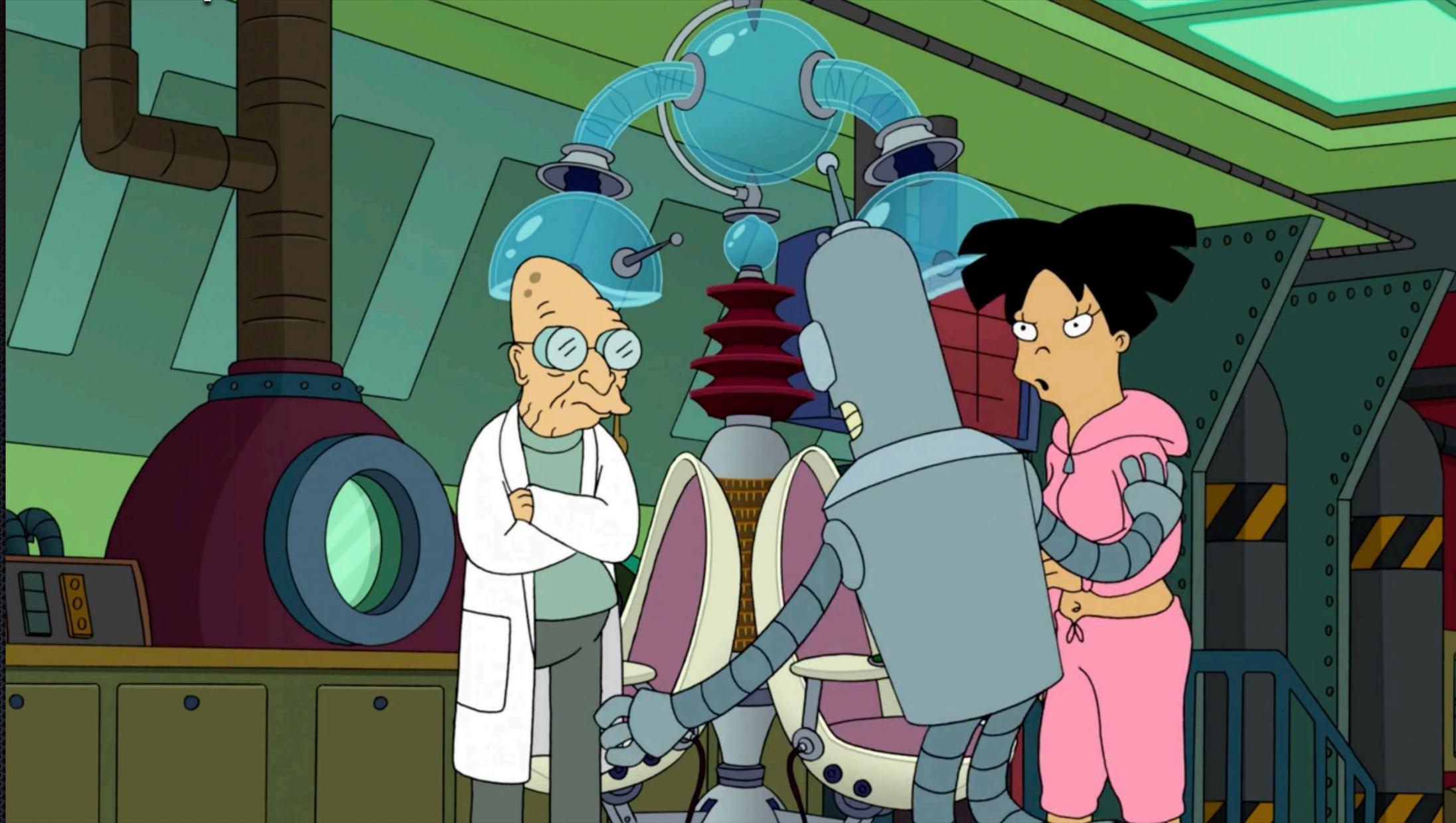


Escribe lo que acaba de ocurrir en el lenguaje de las permutaciones.

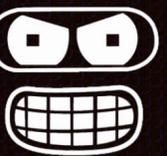
¿Pueden recuperar Amy y el Profesor su cuerpo original? ¿qué operación necesitan hacer en lenguaje matemático? ¿cuál sería?



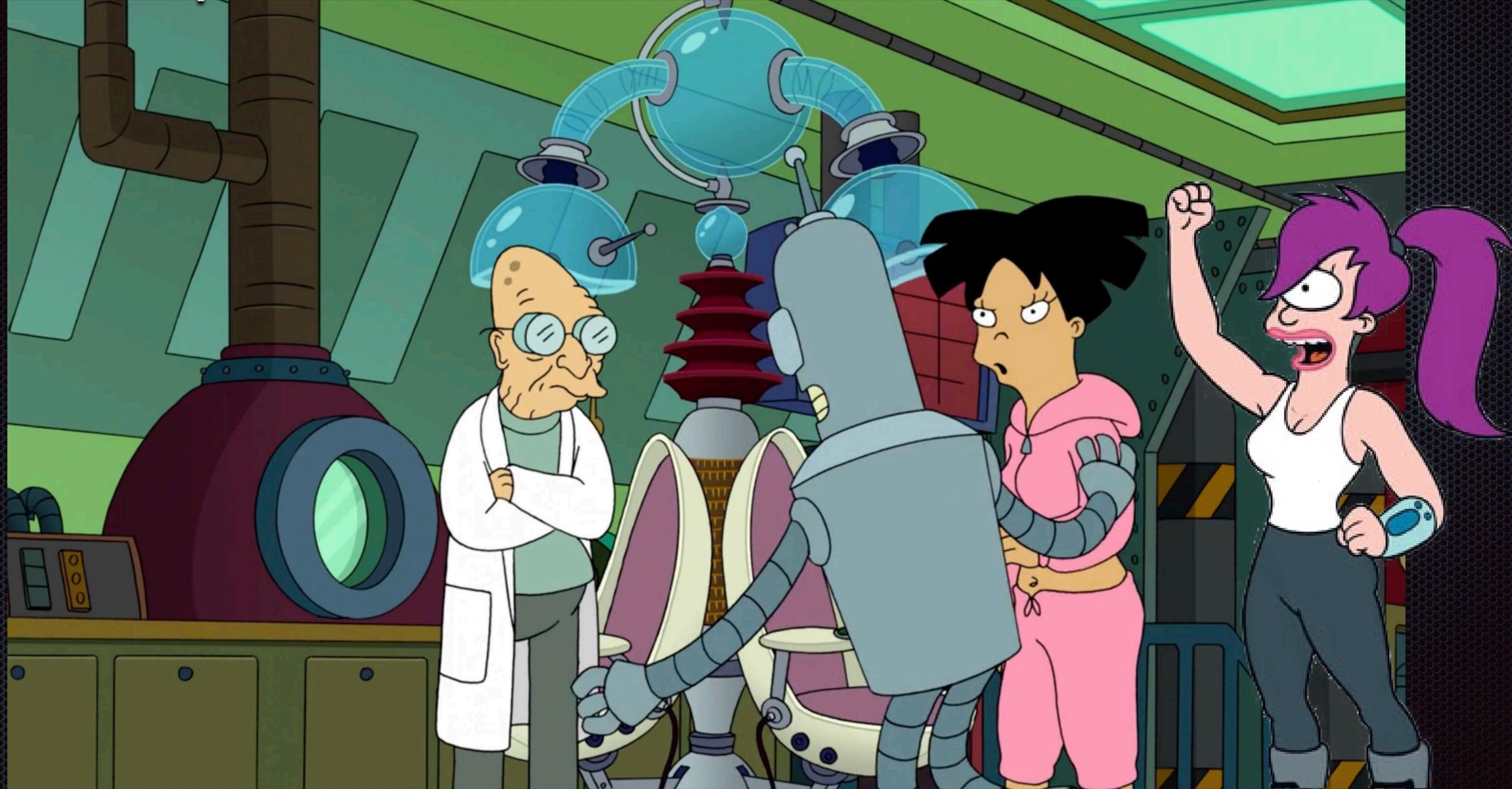
Segunda parte: El vídeo



¿Qué pasa si le piden ayuda a Bender? ¿pueden recuperar los tres su cuerpo original? Es decir, se puede calcular $(1\ 2)^1$ en $\{1, 2, 3\}$ con las restricciones de la máquina.



Segunda parte: El vídeo



¿Y si entra ahora Leela? ¿podrán recuperar su cuerpo original los cuatro? Es decir, en $\{1, 2, 3, 4\}$, ¿existen las inversas de las permutaciones calculadas arriba?



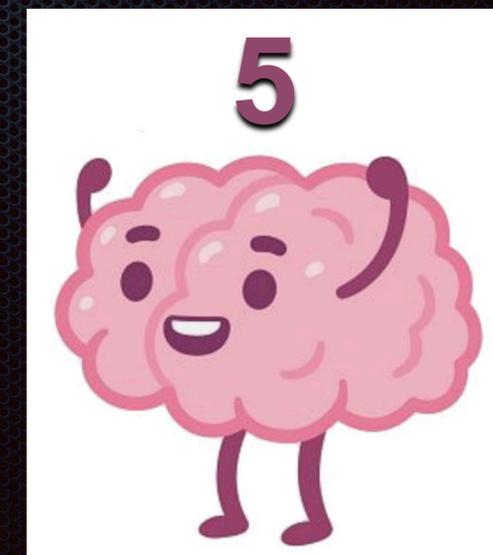
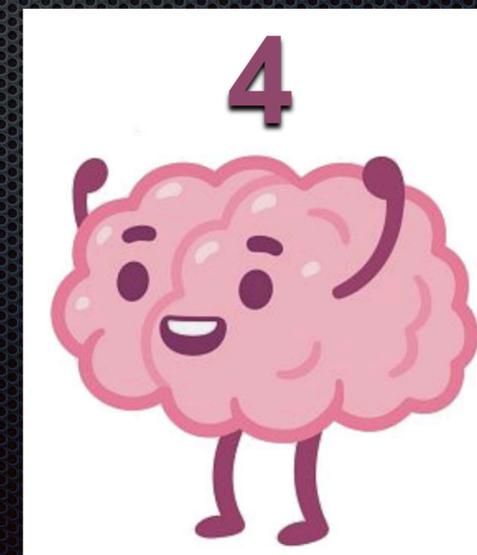
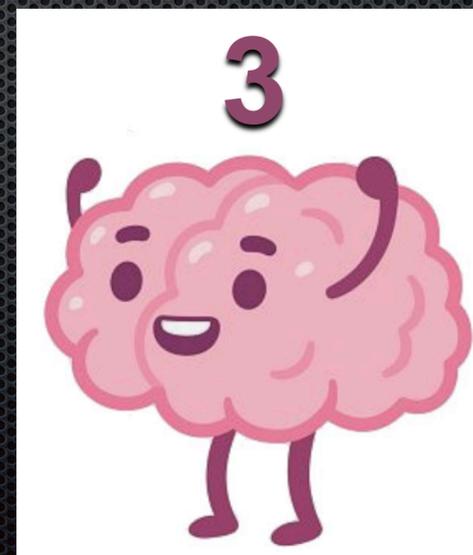
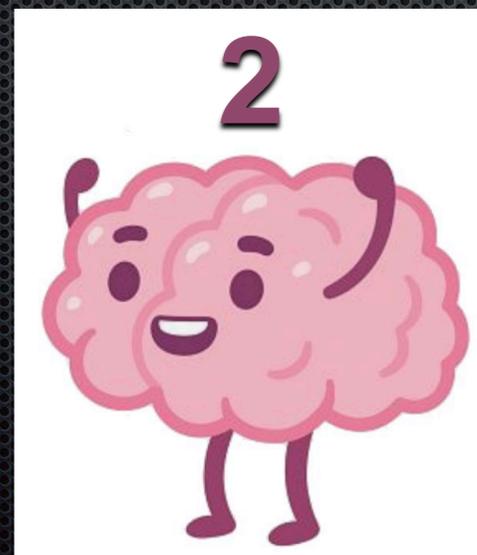
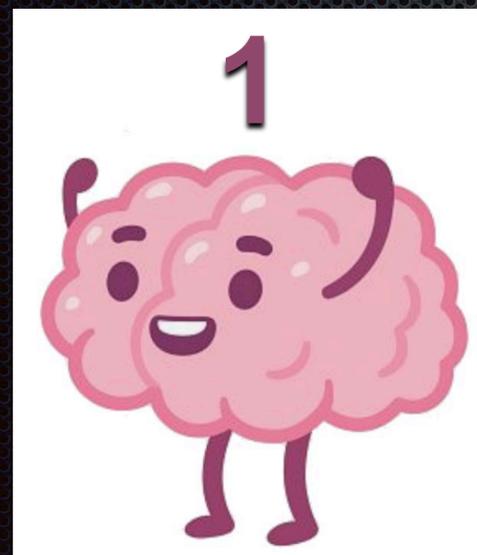
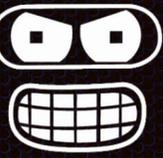
Segunda parte: El vídeo



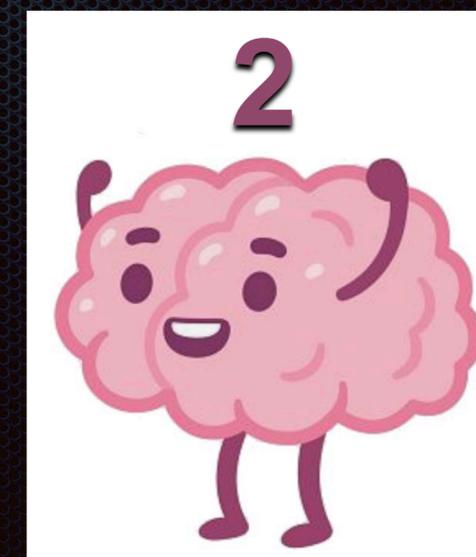
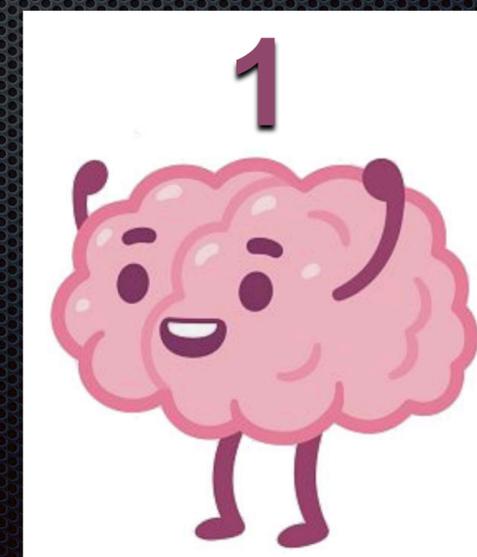
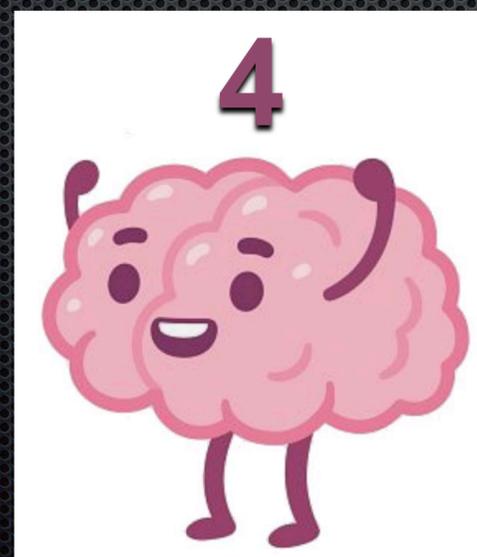
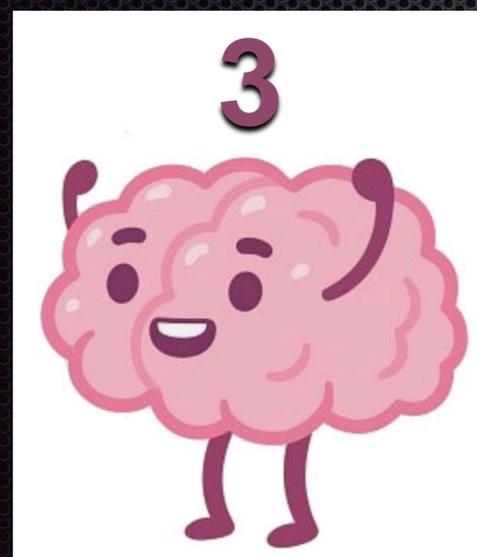
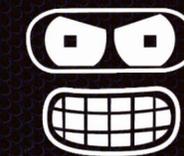
Pregunta: Si en la máquina entran 4 o más personas, ¿existe una manera para que cada uno recupere su cuerpo original?



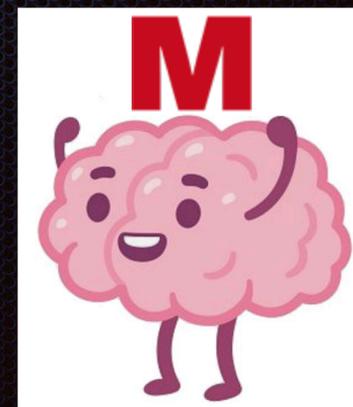
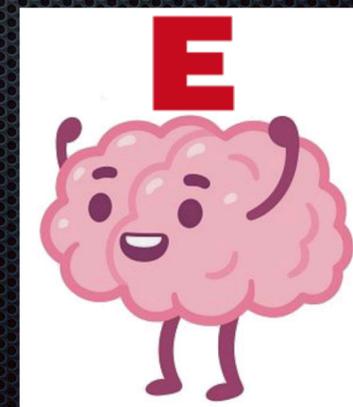
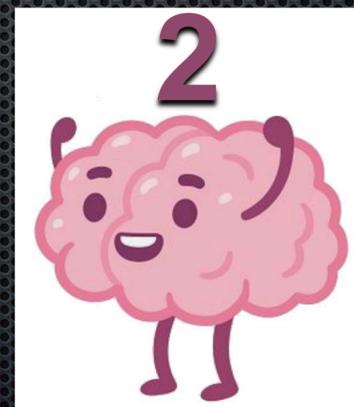
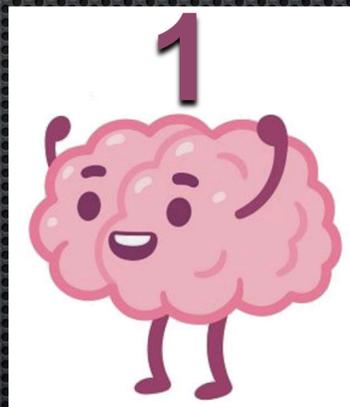
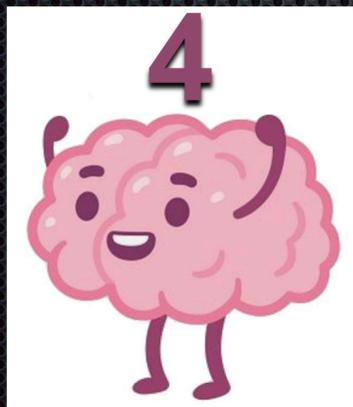
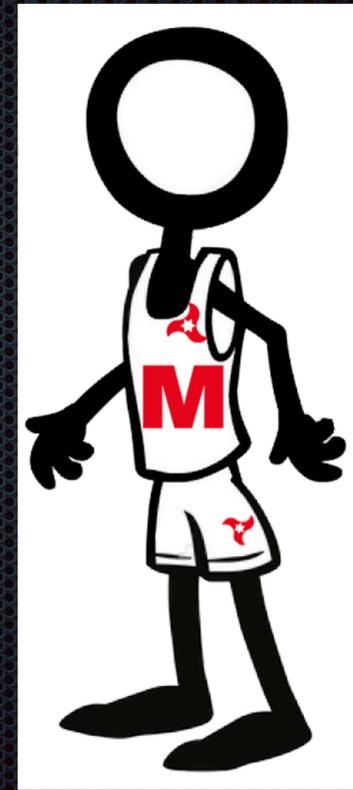
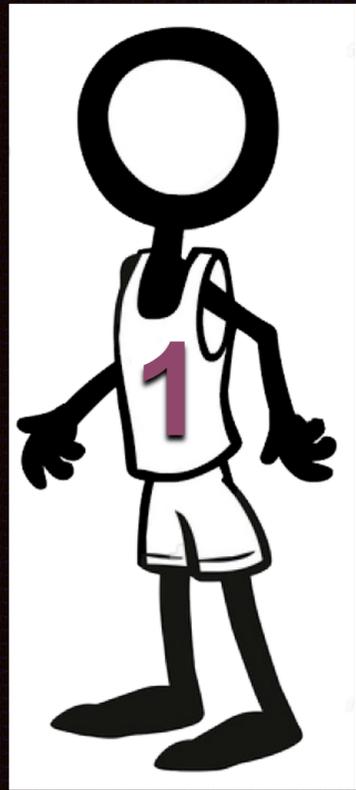
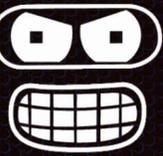
Segunda parte: El juego



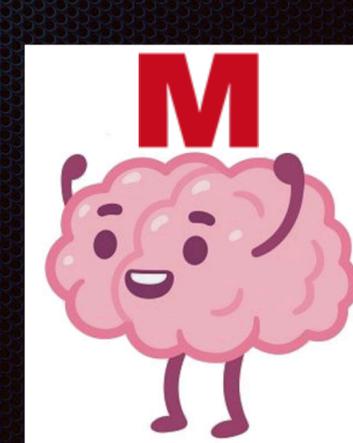
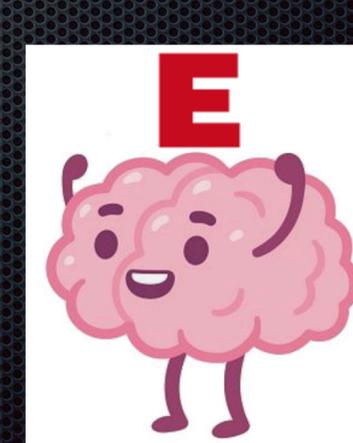
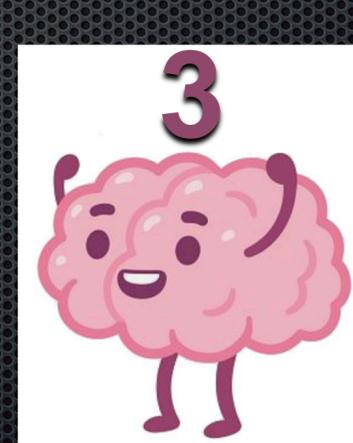
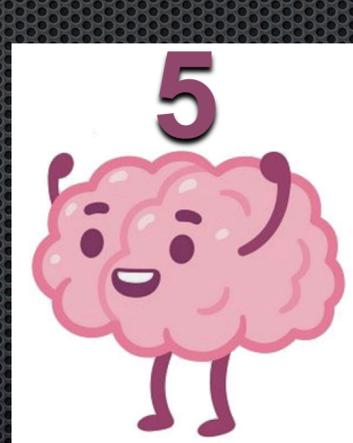
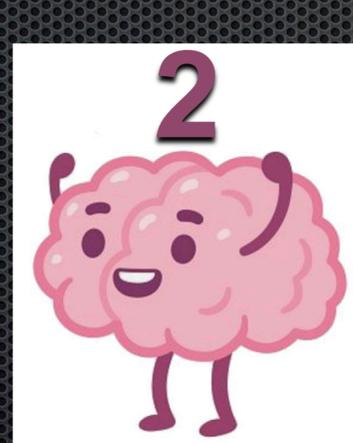
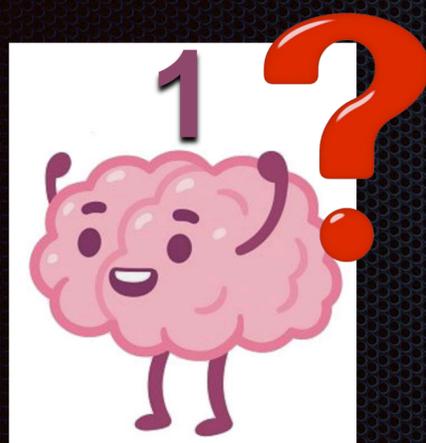
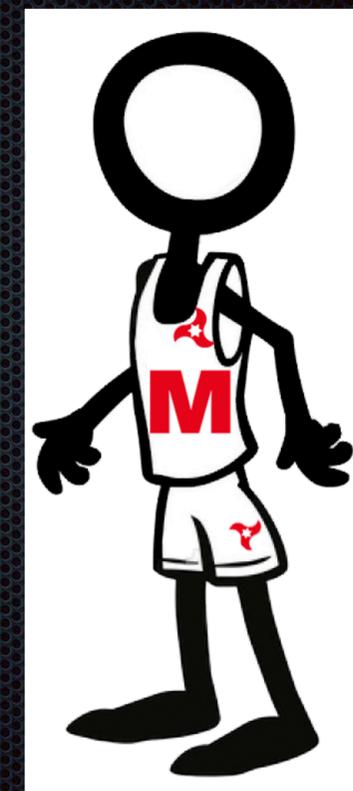
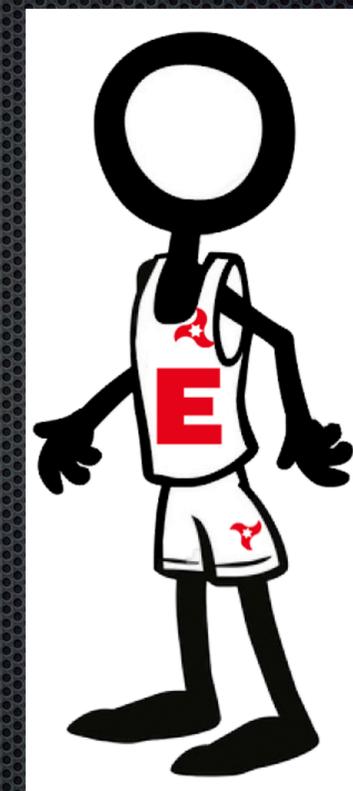
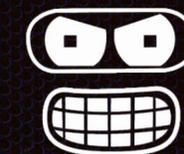
Segunda parte: El juego



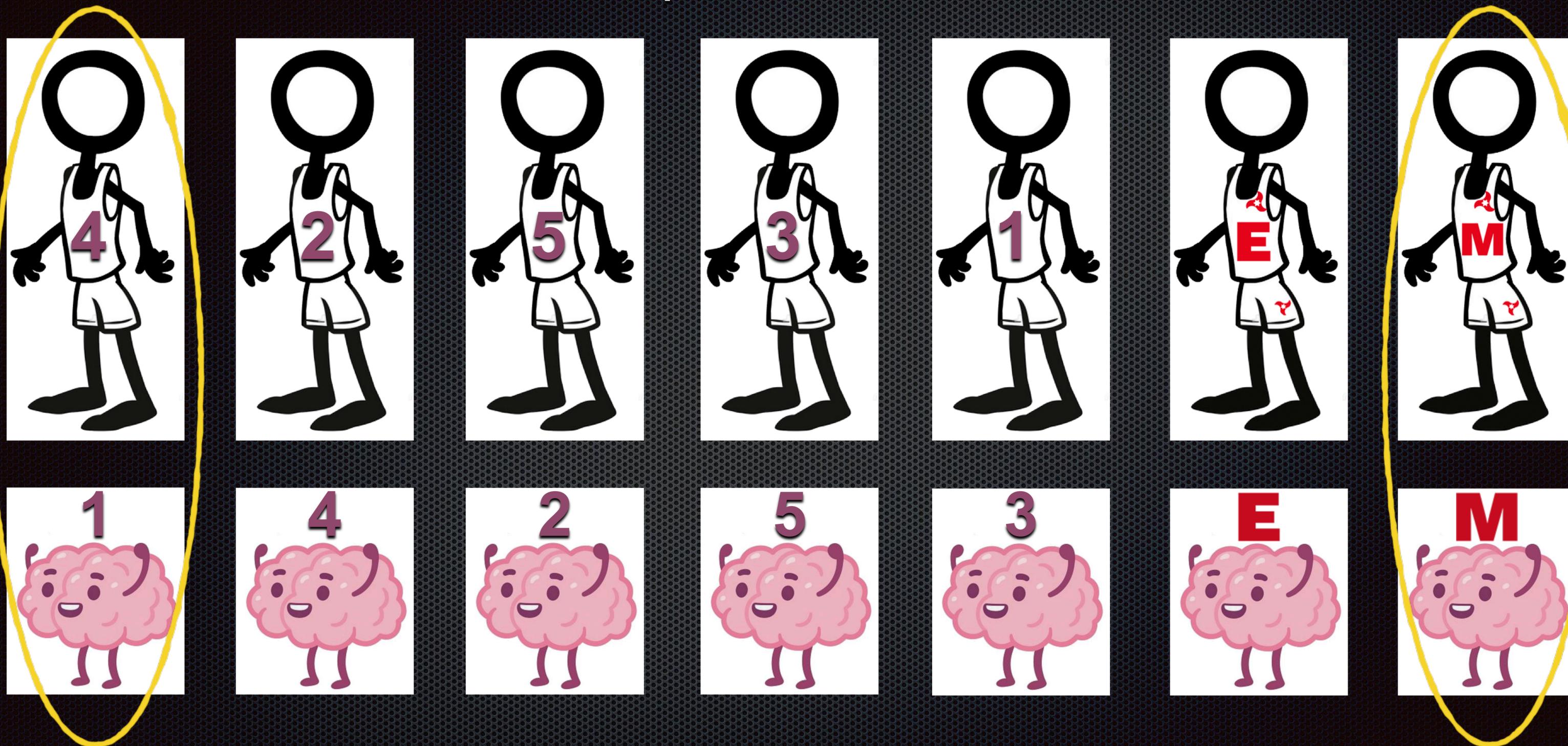
La solución: fichajes estrella



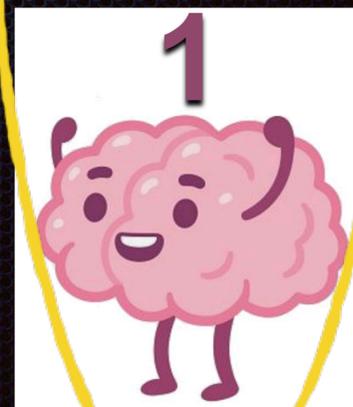
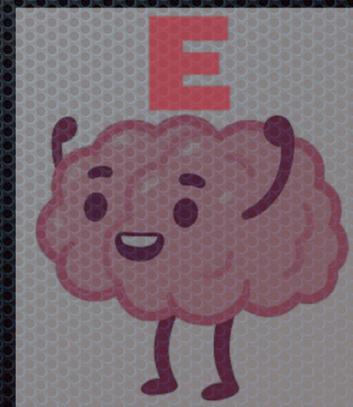
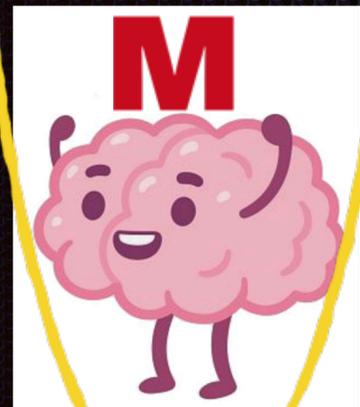
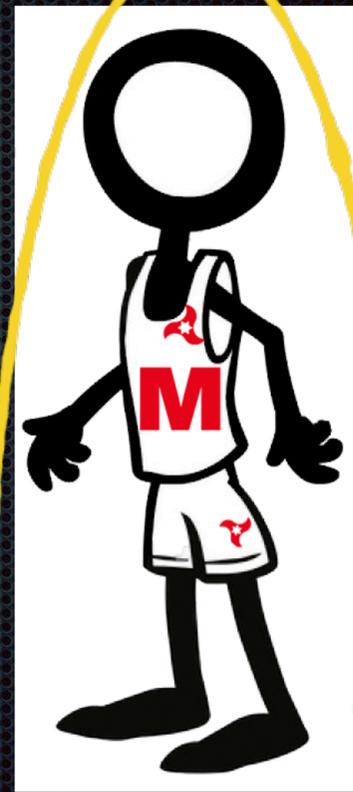
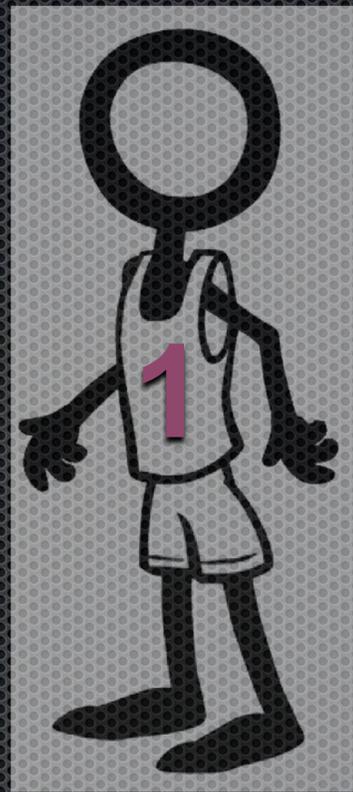
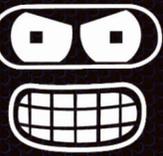
La solución: ordenamos



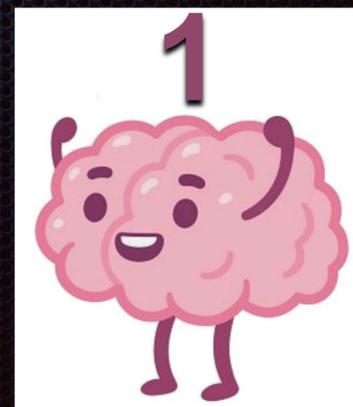
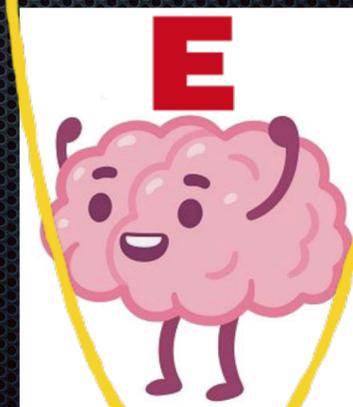
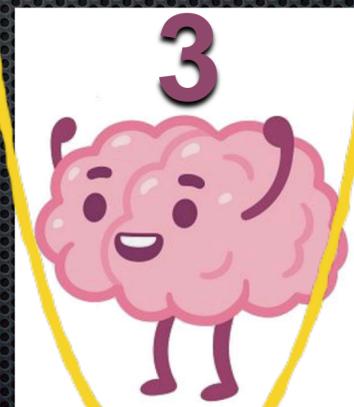
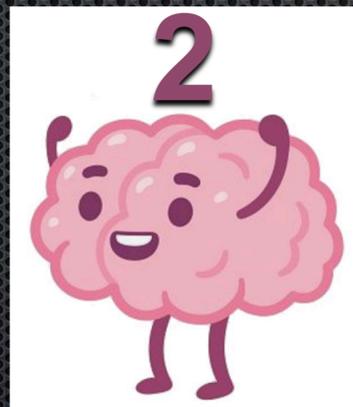
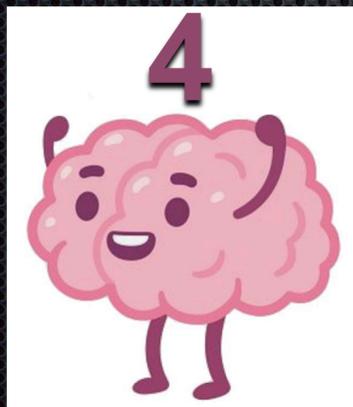
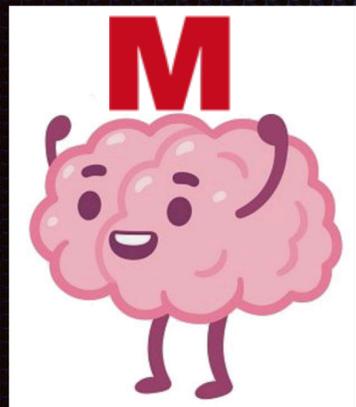
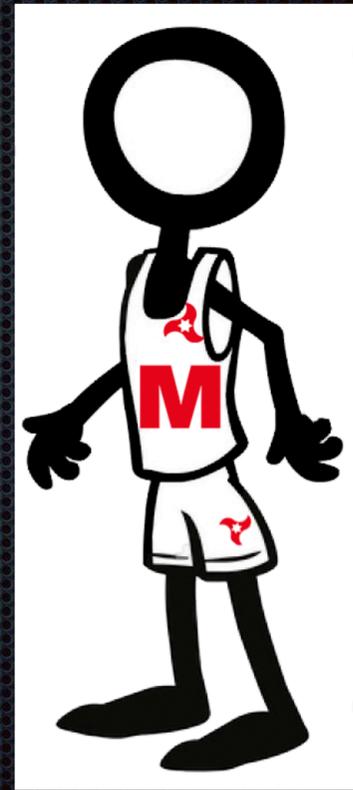
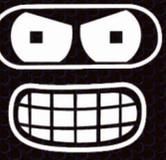
La solución: M a por el cerebro del 1



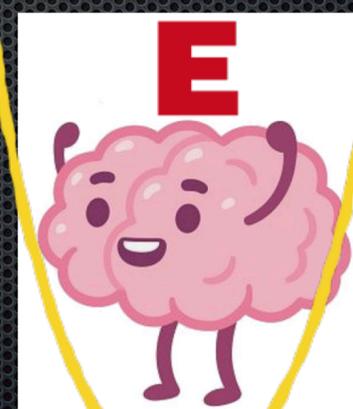
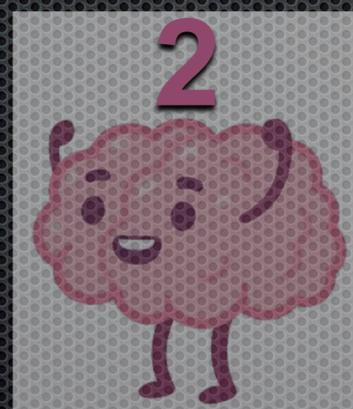
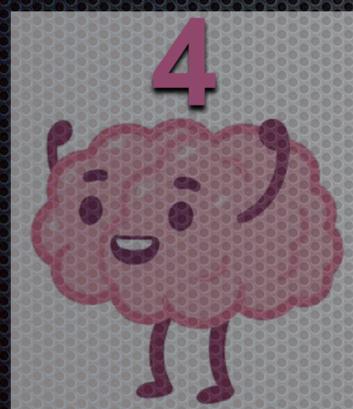
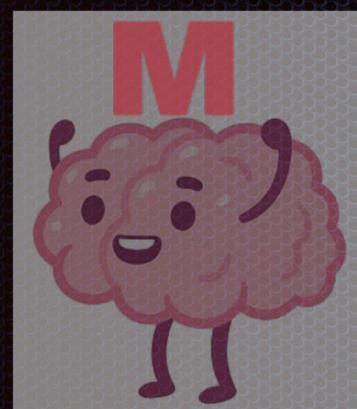
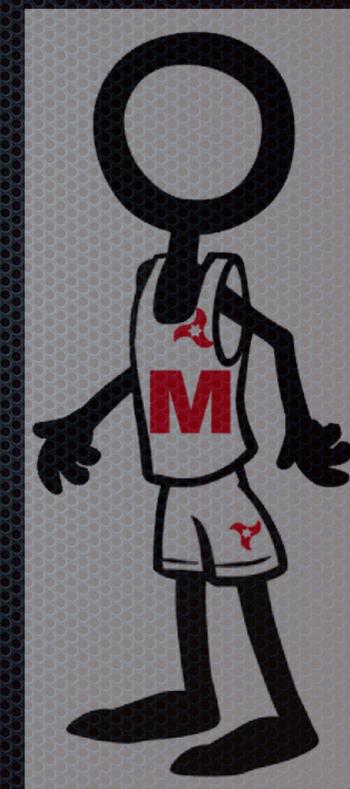
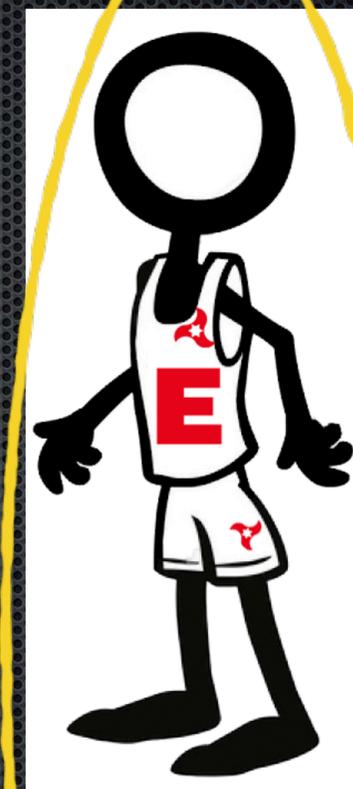
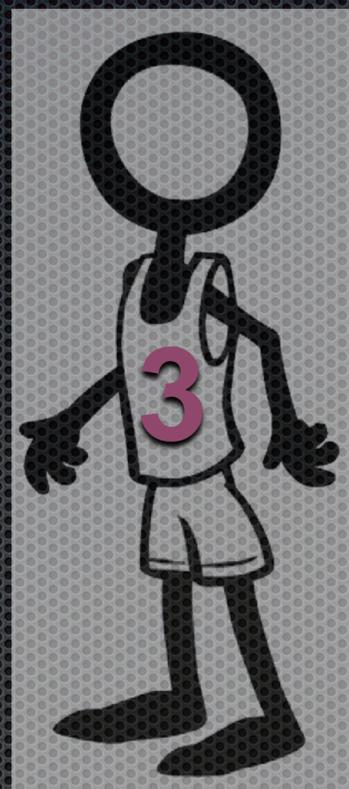
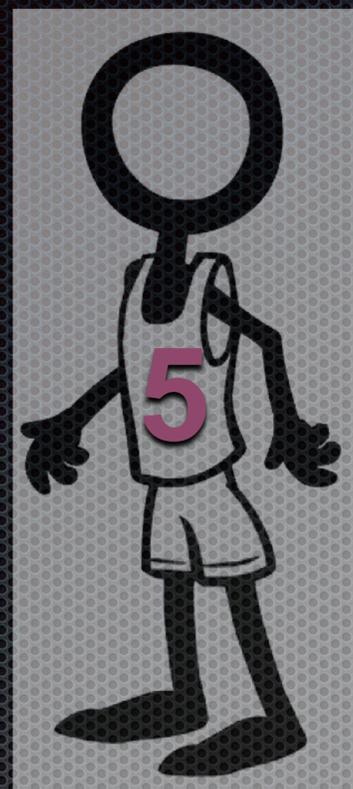
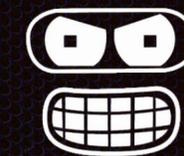
La solución: M a por el cerebro del 1



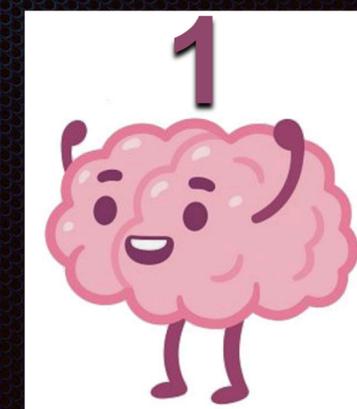
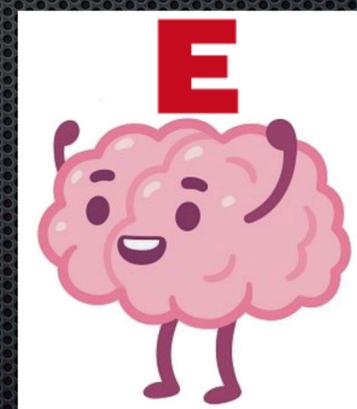
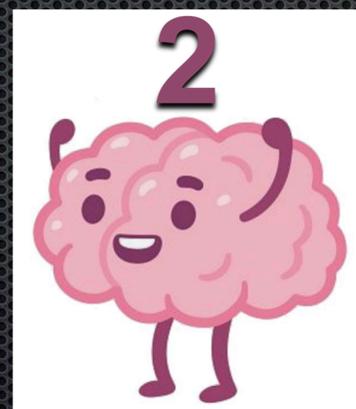
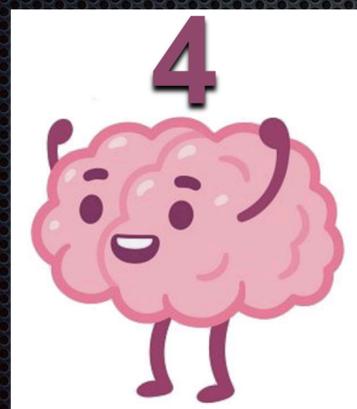
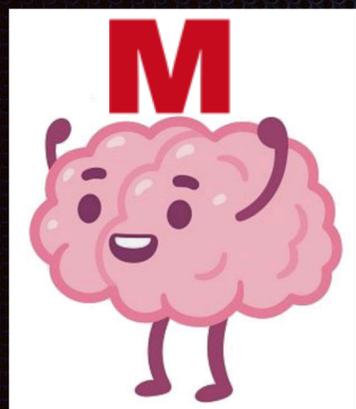
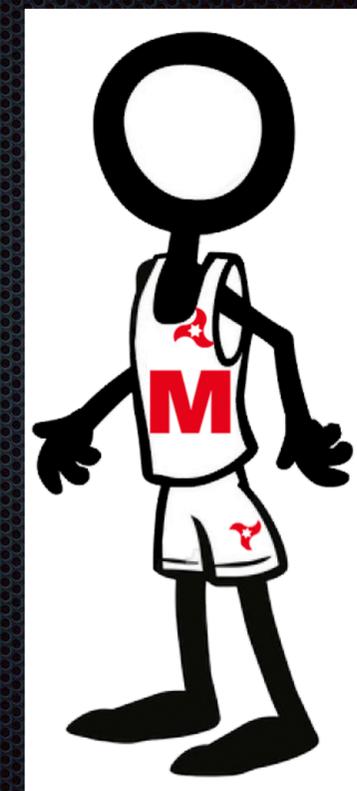
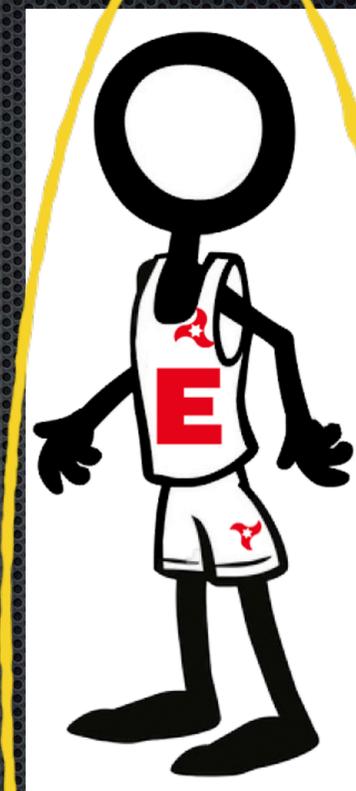
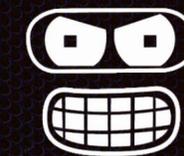
La solución: E pivotando en orden



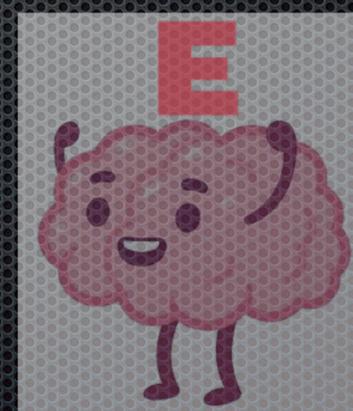
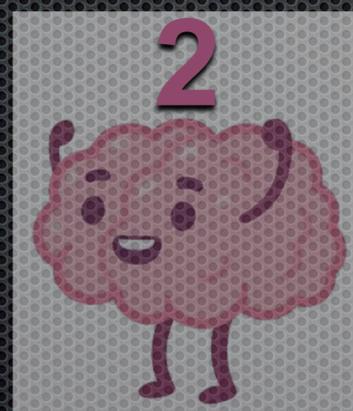
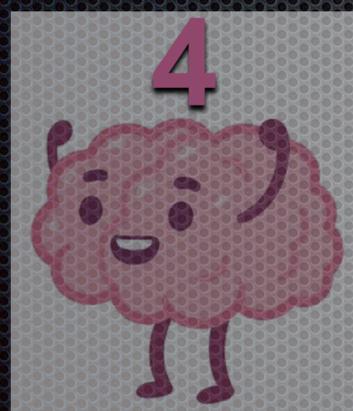
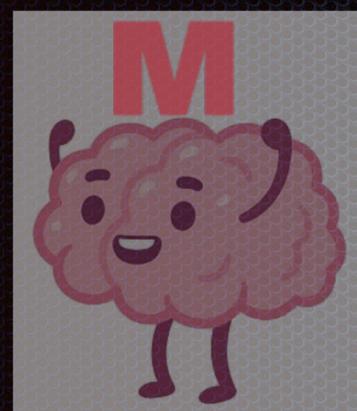
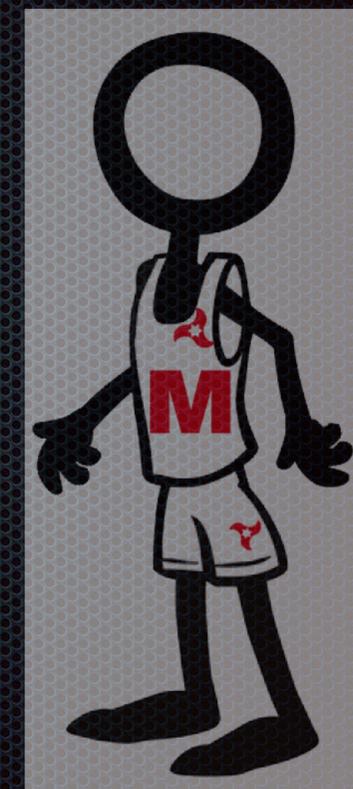
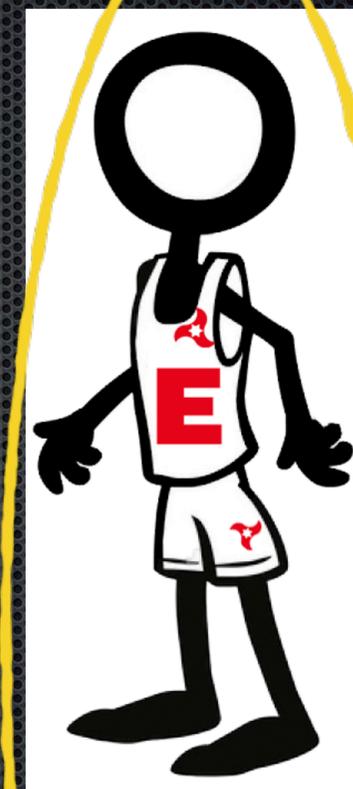
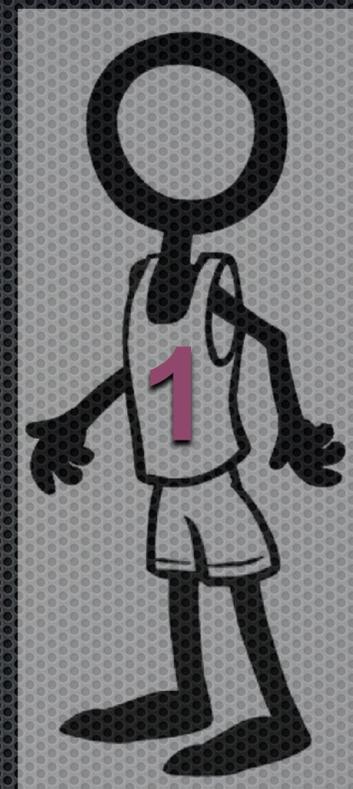
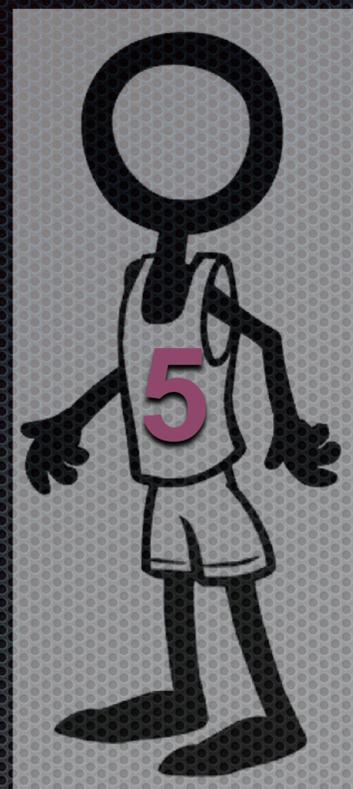
La solución: E pivotando en orden



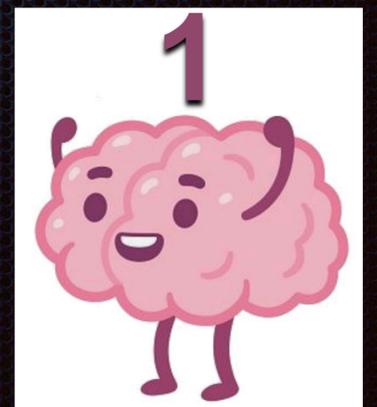
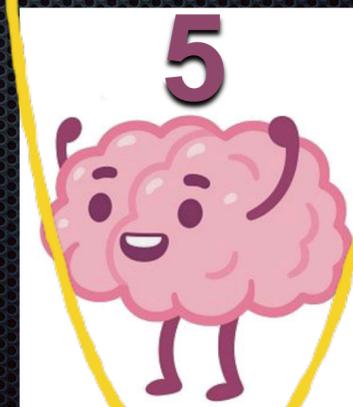
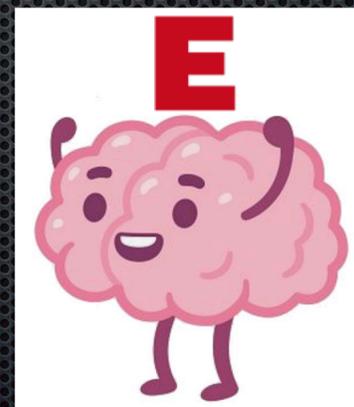
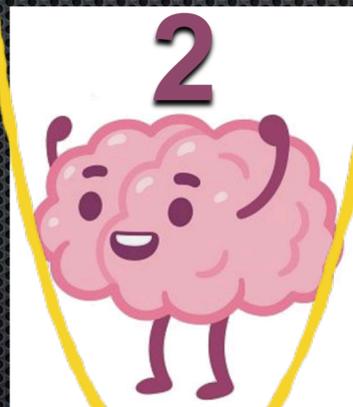
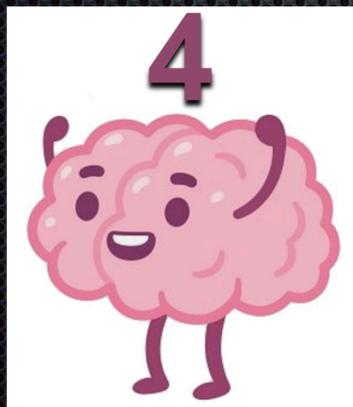
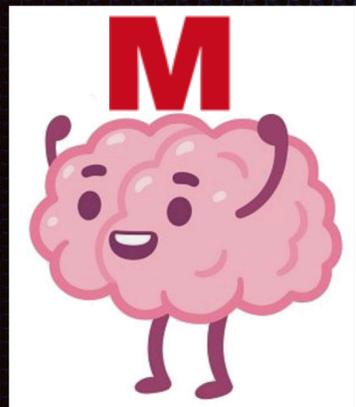
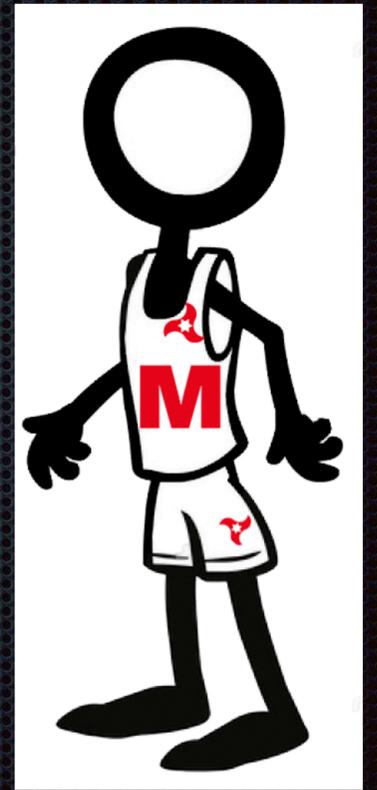
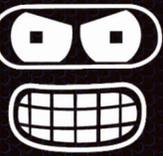
La solución: E pivotando en orden



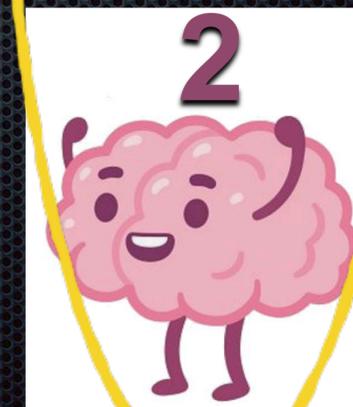
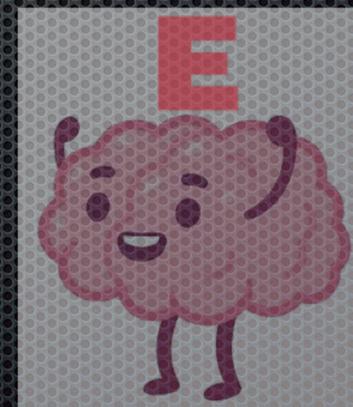
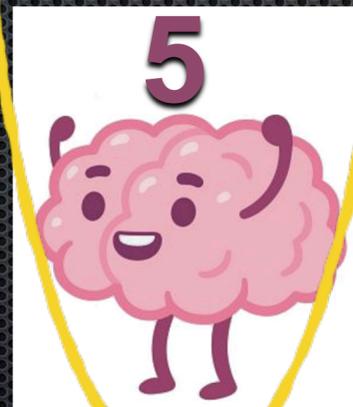
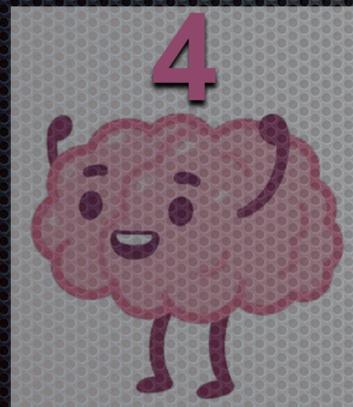
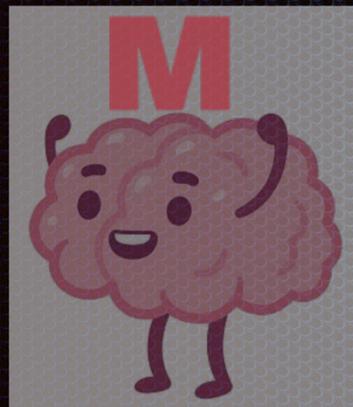
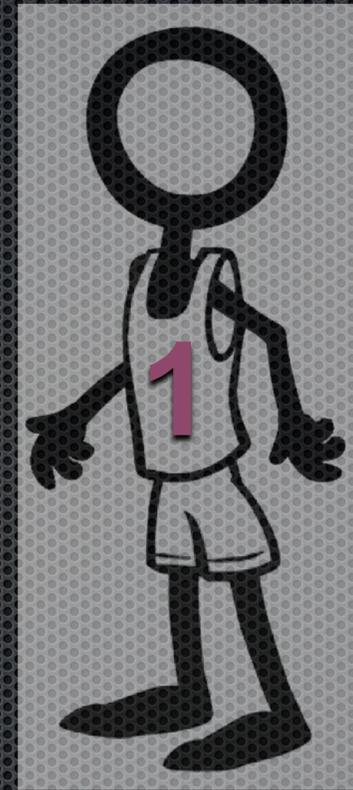
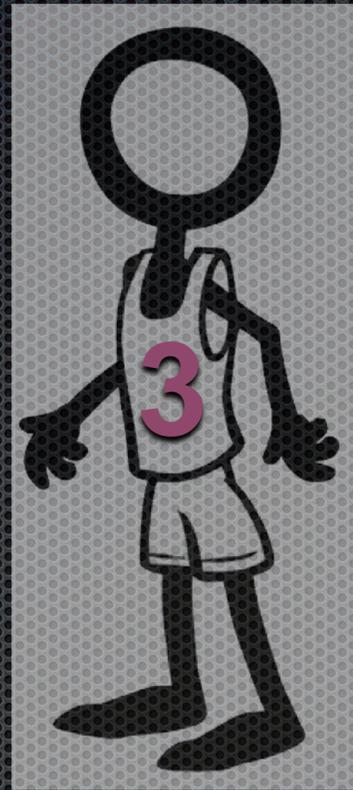
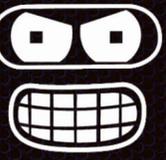
La solución: E pivotando en orden



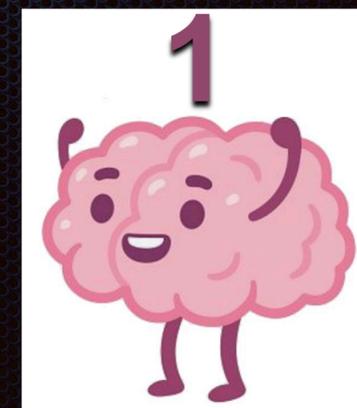
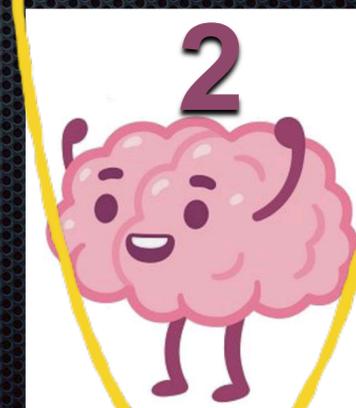
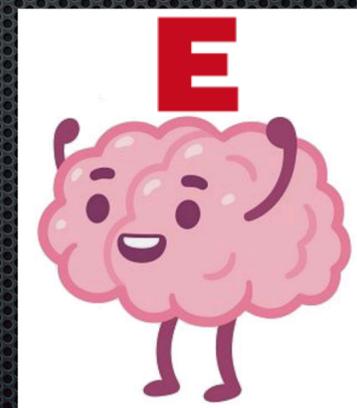
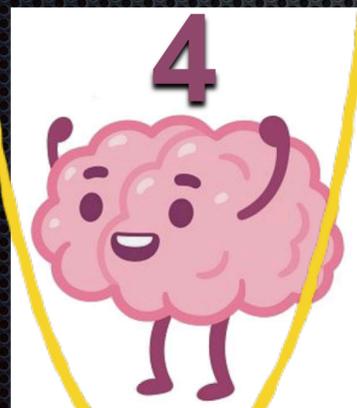
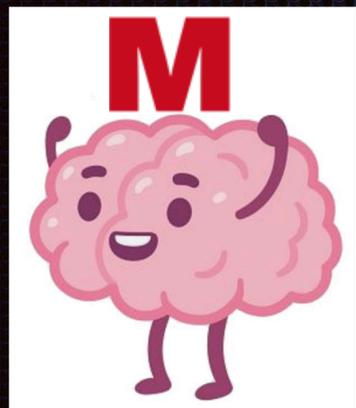
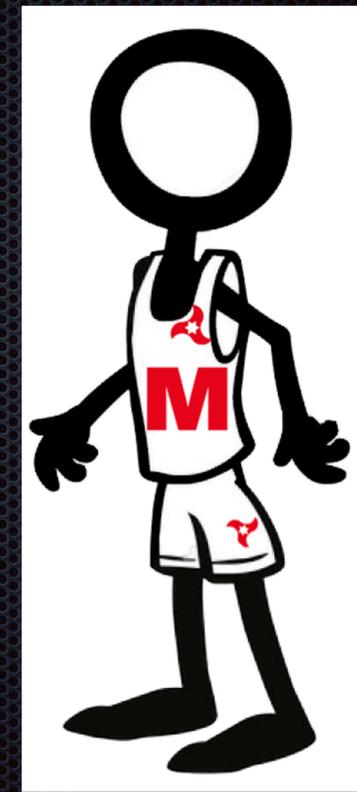
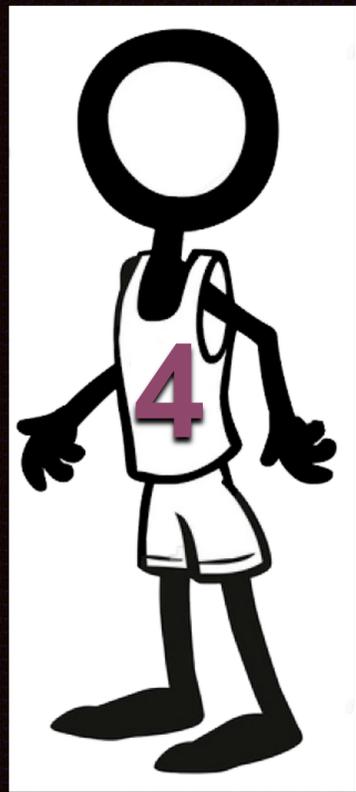
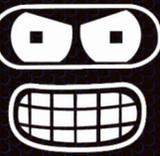
La solución: E pivotando en orden



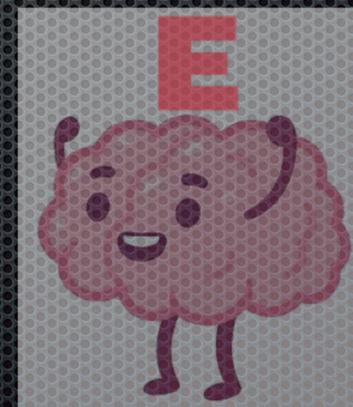
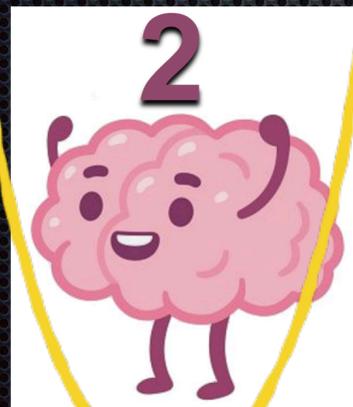
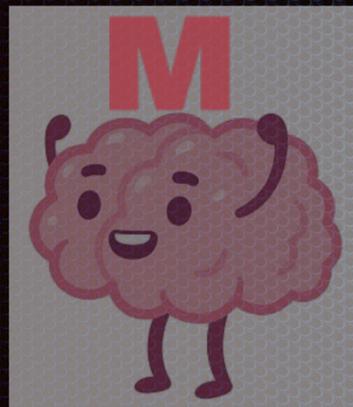
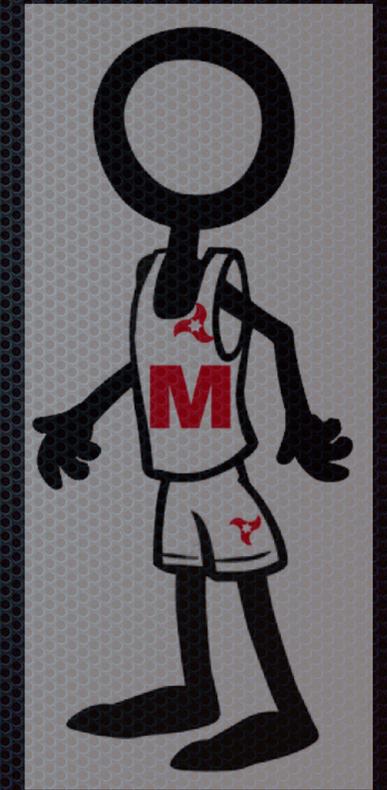
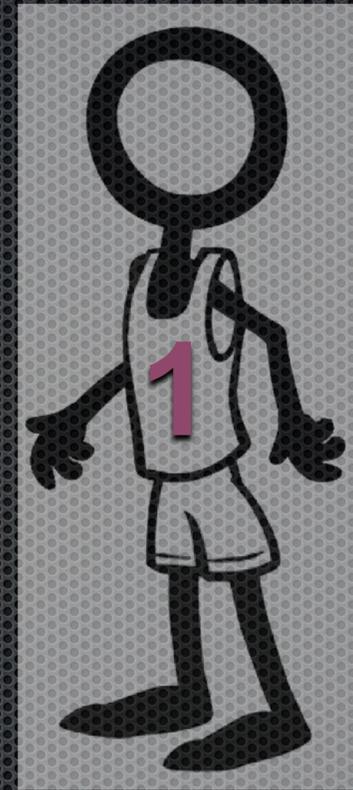
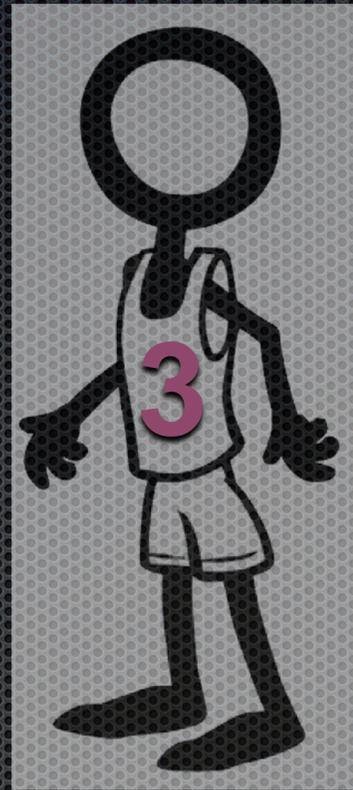
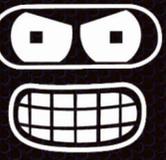
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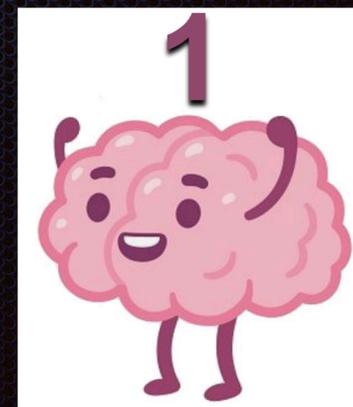
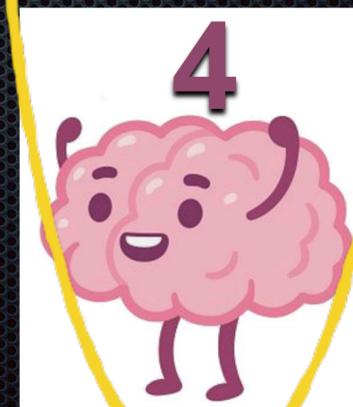
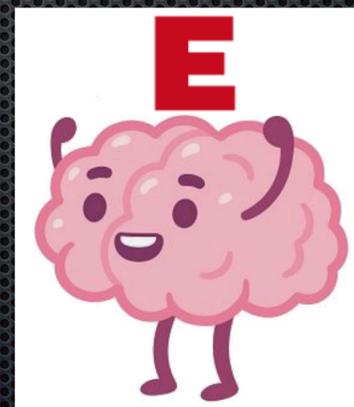
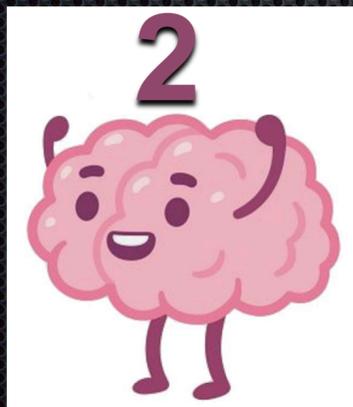
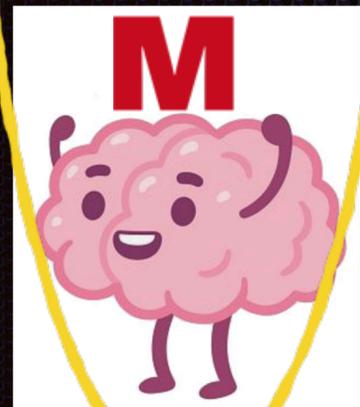
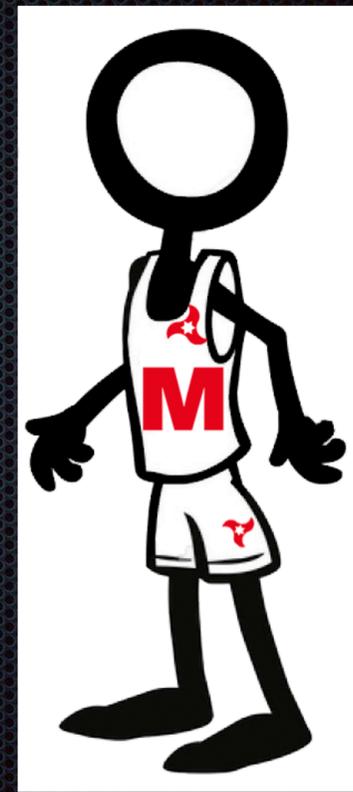
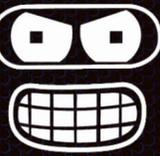
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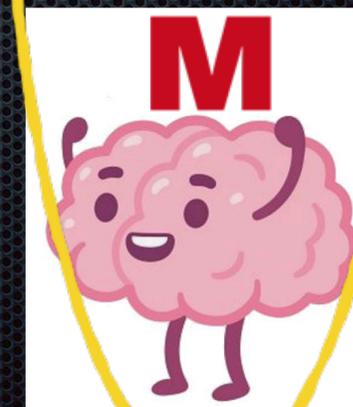
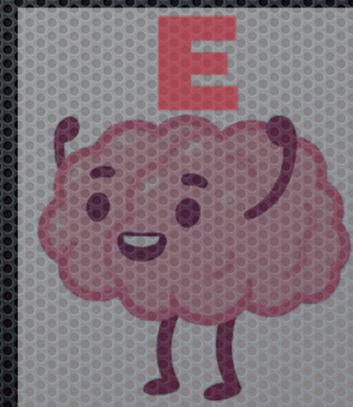
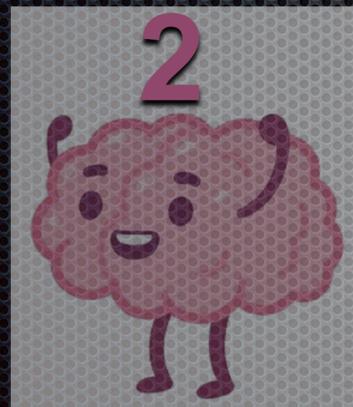
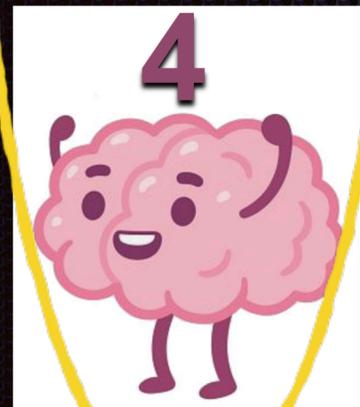
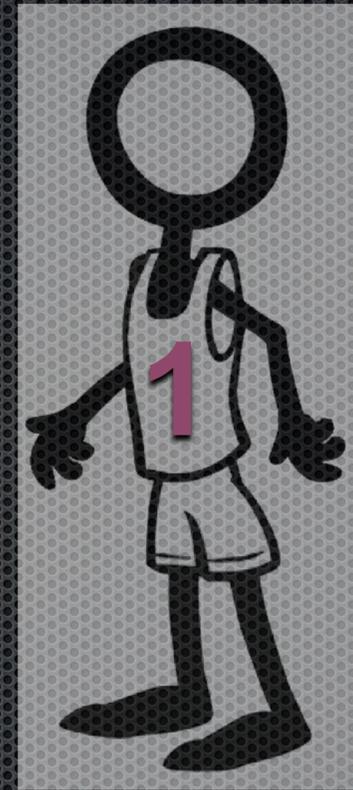
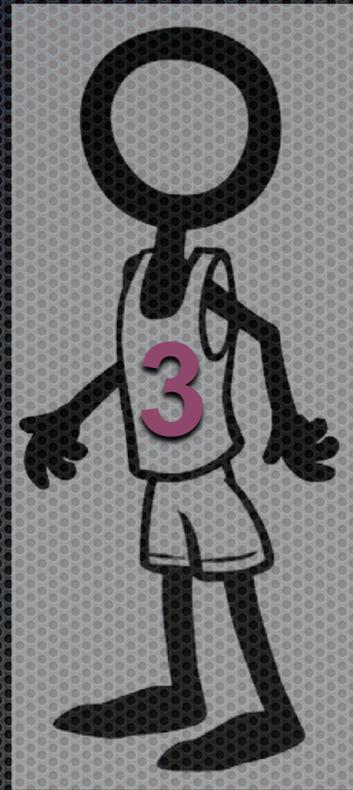
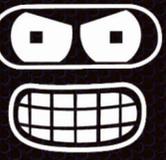
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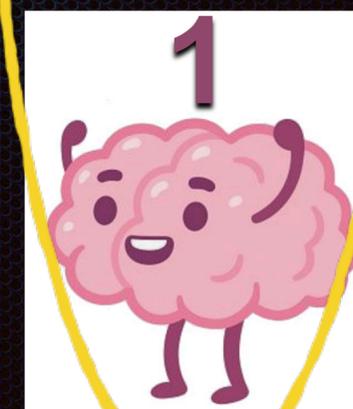
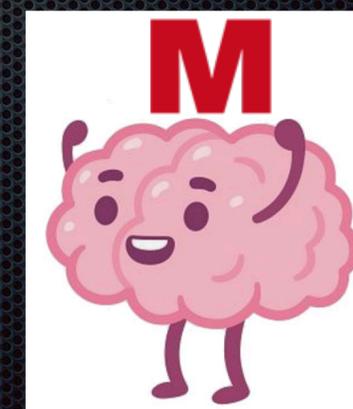
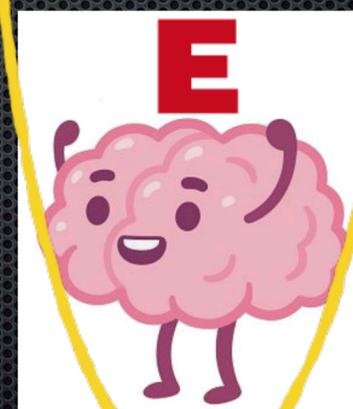
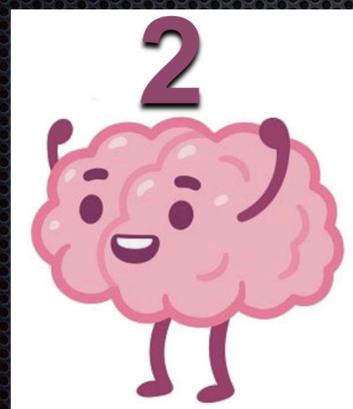
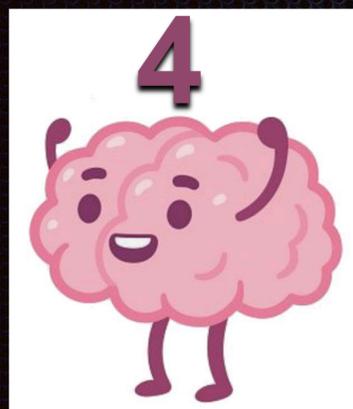
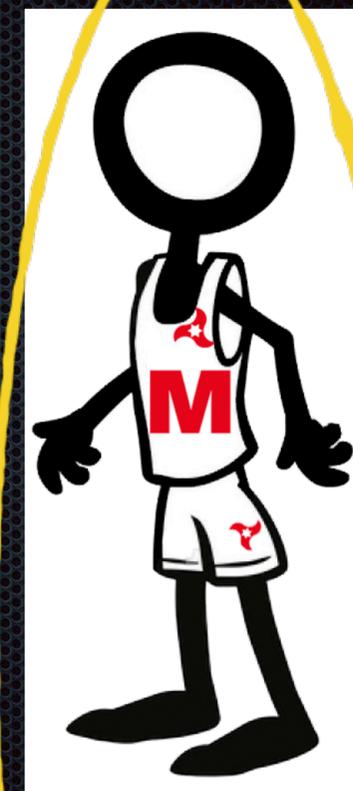
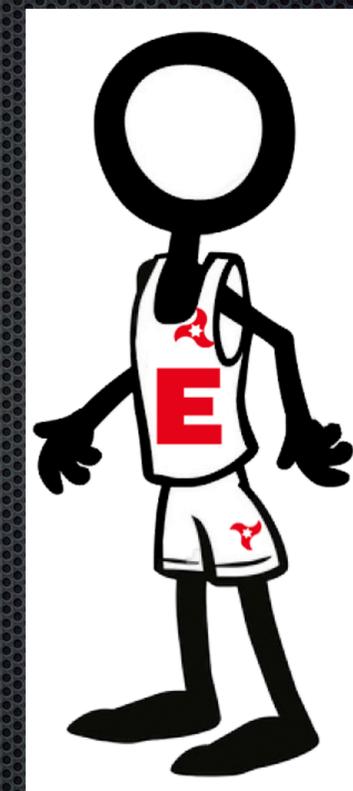
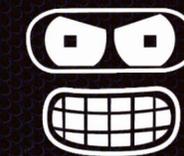
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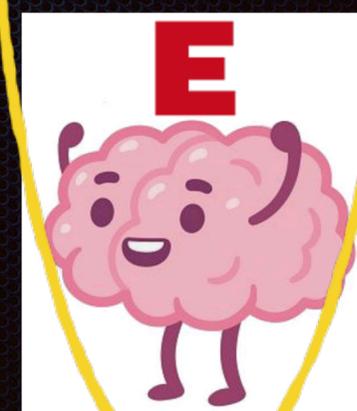
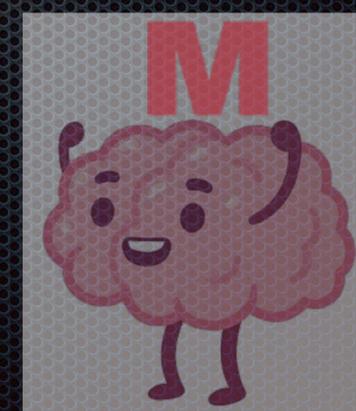
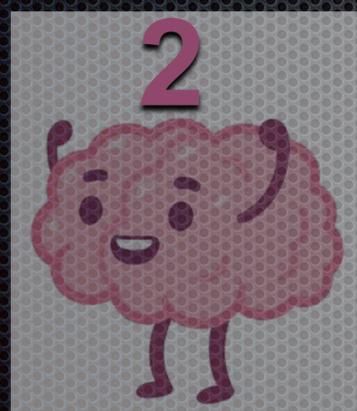
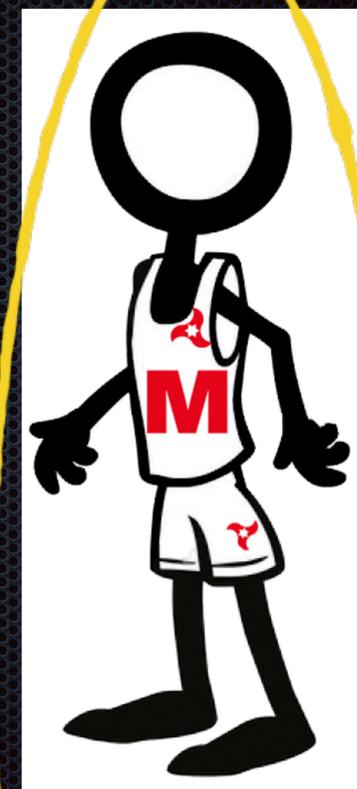
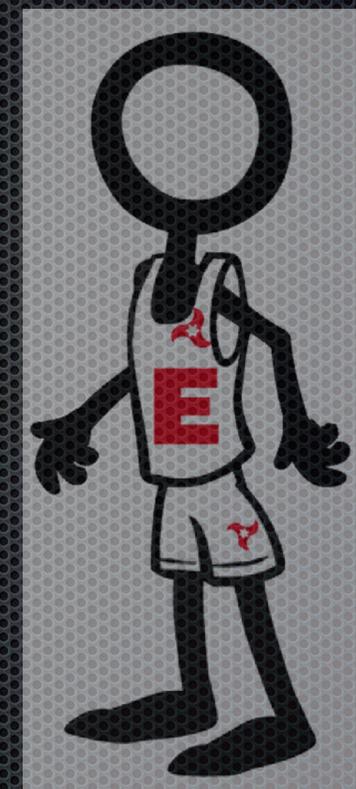
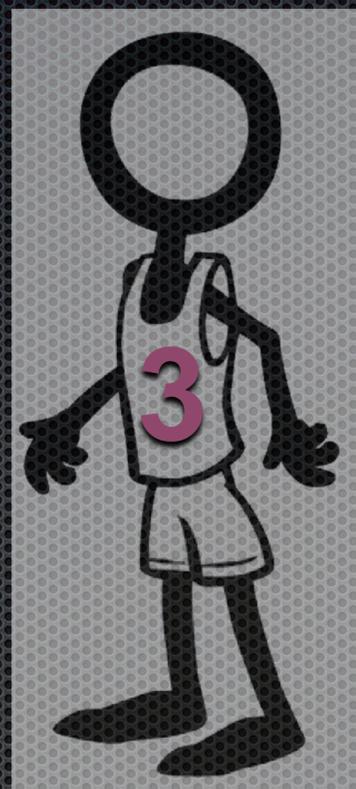
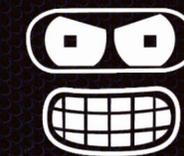
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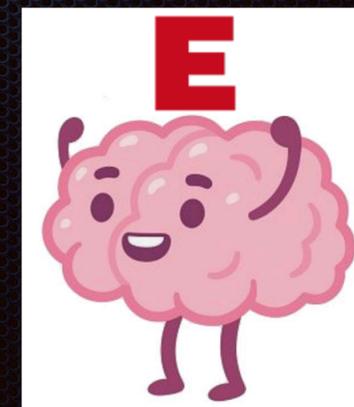
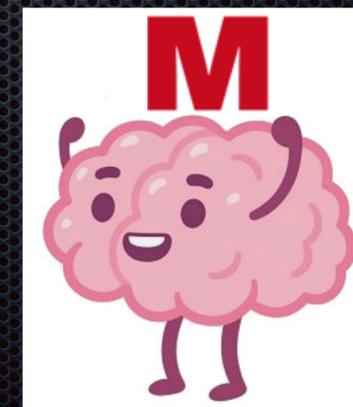
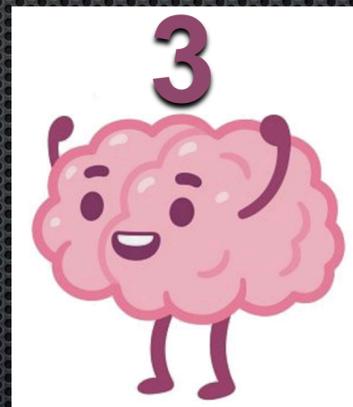
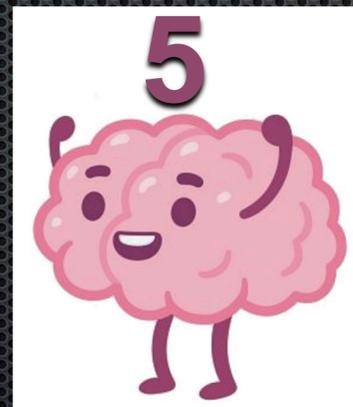
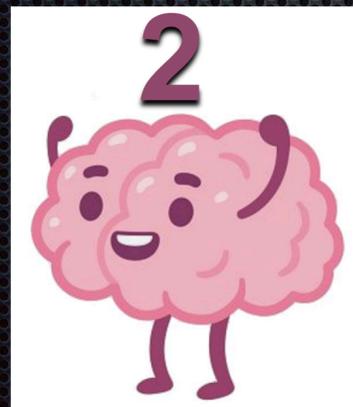
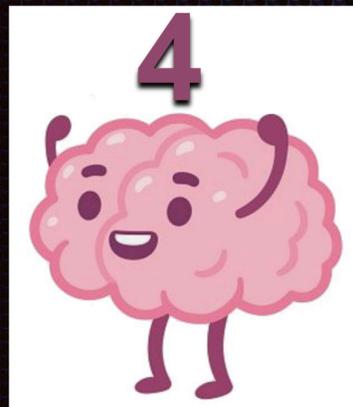
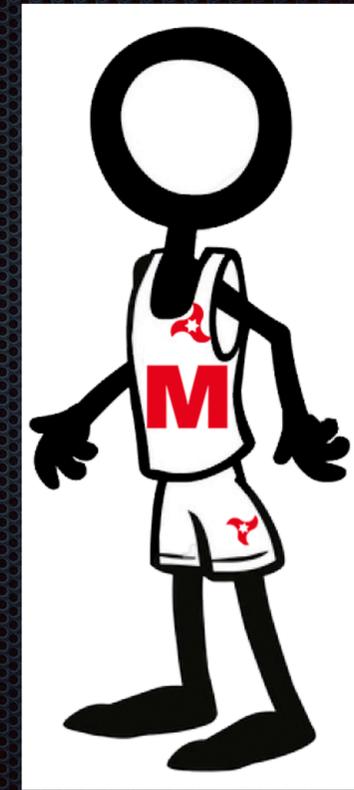
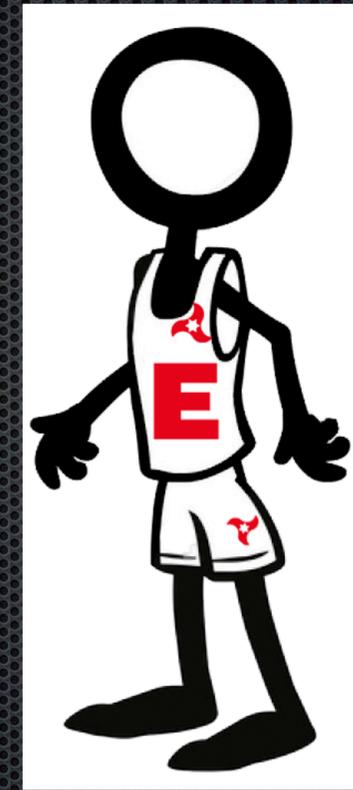
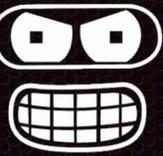
La solución: M devuelve el cerebro a 1



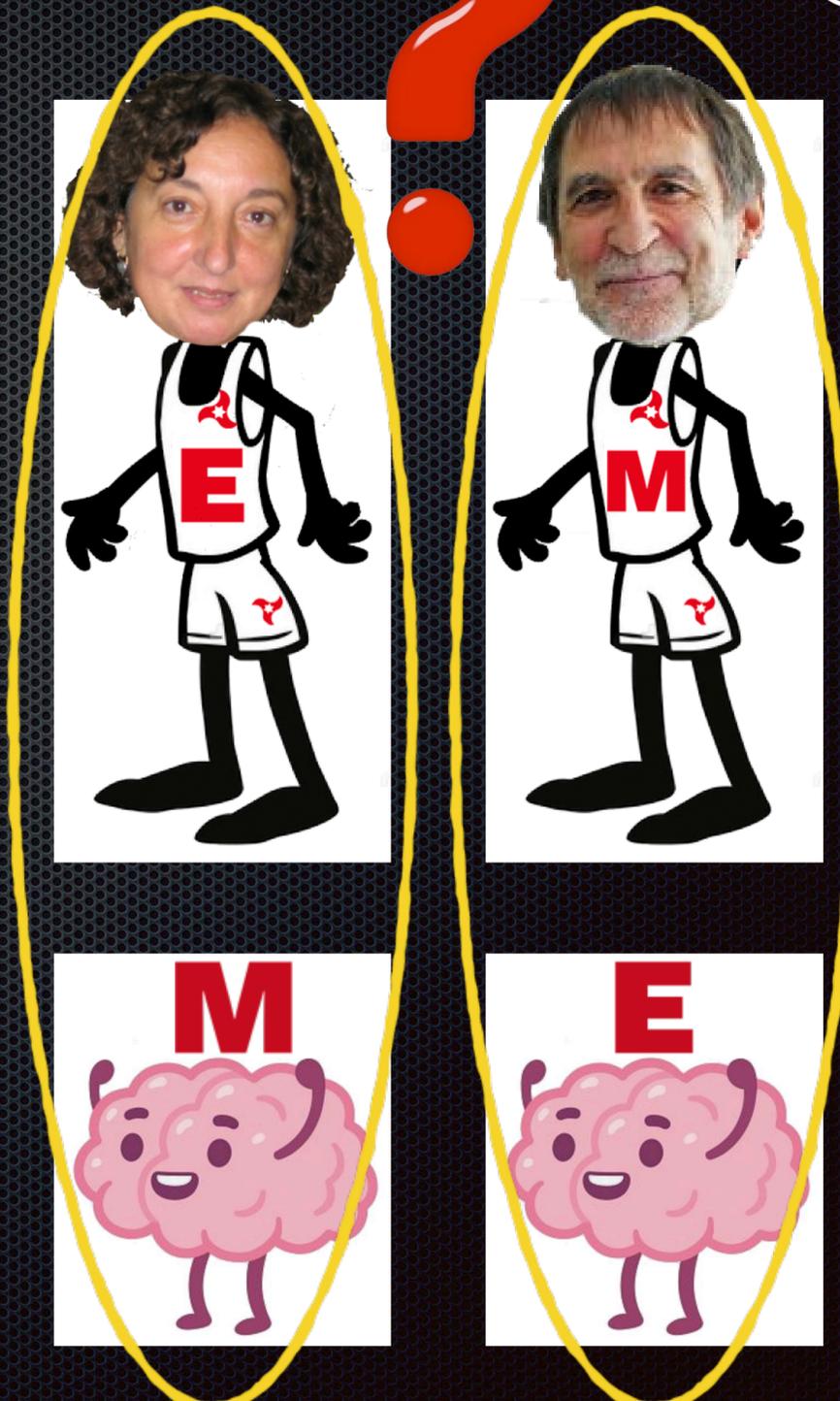
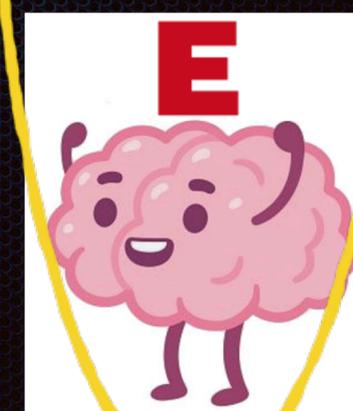
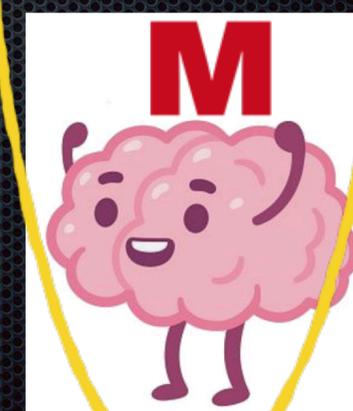
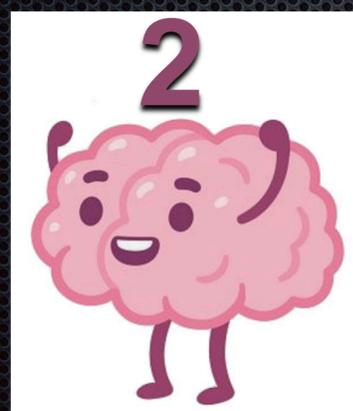
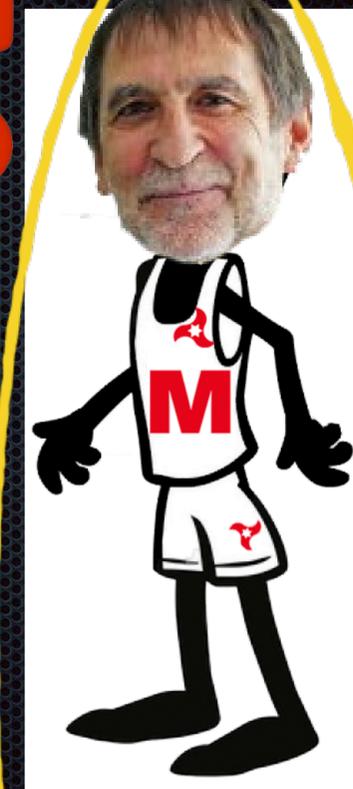
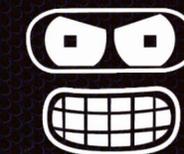
La solución: M devuelve el cerebro a 1



Segunda parte: La solución



Segunda parte: La solución



El Teorema

No importa cómo un grupo de gente haya intercambiado sus mentes y sus cuerpos, siempre es posible que cada persona recupere su cuerpo usando como mucho dos personas extra.

First, let π be some k -cycle on $[n] = \{1 \dots n\}$: WLOG write

$$\pi = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n \\ 2 & 3 & \dots & 1 & k+1 & \dots & n \end{pmatrix}$$

Let $\langle a, b \rangle$ represent the transposition that switches the contents of a and b .
By hypothesis π is generated by DISTINCT switches on $[n]$.

Introduce two "new bodies" $\{x, y\}$ and write $\pi^* = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n & x & y \\ 2 & 3 & \dots & 1 & k+1 & \dots & n & x & y \end{pmatrix}$

For any $i = 1, \dots, k$ let σ be the (L-to-R) series of switches

$$\sigma = (\langle x, 1 \rangle \langle x, 2 \rangle \dots \langle x, i \rangle) (\langle y, i+1 \rangle \langle y, i+2 \rangle \dots \langle y, k \rangle) (\langle x, i+1 \rangle) (\langle y, 1 \rangle).$$

Note each switch exchanges an element of $[n]$ with one of $\{x, y\}$, so they're all distinct from the switches within $[n]$ that generated π , and also from $\langle x, y \rangle$. By routine verification,

$$\pi^* \sigma = \begin{pmatrix} 1 & 2 & \dots & n & x & y \\ 1 & 2 & \dots & n & y & x \end{pmatrix} \quad \text{i.e., } \sigma \text{ inverts the } k\text{-cycle and leaves } x \text{ and } y \text{ switched (without performing } \langle x, y \rangle \text{).}$$

NOW let π be an ARBITRARY permutation on $[n]$: it consists of disjoint (nontrivial) cycles, and each can be inverted as above in sequence, after which x and y can be switched if necessary via $\langle x, y \rangle$, as was desired.



GRACIAS



La solución en términos de permutaciones

Comentarios: los nuevos cambios de cuerpos se componen por la izquierda (leyendo de izda a derecha la composición, como siempre).

En el ejemplo (1 3 5 2 4)

Empezamos con el pivote (4 M) (aquí 4 es el último)

$$(4 M) (1 3 5 2 4) = (1 3 5 2 4 M)$$

Y ahora empezaremos a pivotar con E, con los números de izda a dcha

$$(1 E) (1 3 5 2 4 M) = (1 E 3 5 2 4 M); (3 E) (1 E 3 5 2 4 M) = (1 E 5 2 4 M)$$

$$(5 E) (1 E 5 2 4 M) = (1 E 2 4 M); (2 E) (1 E 2 4 M) = (1 E 4 M);$$

$$(4 E) (1 E 4 M) = (1 E M)$$

Ahora volvemos al primer pivote que entra con el que queda (el primero de la lista)

$$(1 M) (1 E M) = (E M) \text{ y ya los pivotes.}$$

